

Category

has • objects
• morphisms

satisfies • composition
• associativity
• identity

Terminal object 1

$$\forall A \in \text{Ob } \mathcal{C}, \exists ! 1_A: A \rightarrow 1.$$

Examples of Categories

- sets A • sets
→ • functions between sets
- sets_{fin} A • finite sets
→ • functions between sets
- poset A • partially ordered set with binary relation
 $a \leq_A b$ such that we have
 - reflexivity
 - transitivity
 - antisymmetry→ • monotone funkcije

• Rel A. sets

→ • binary relations

$$(R: A \rightarrow B = \{(a, b) \in A \times B; a \leq_R b\})$$

• composition:

$$S \circ R = \{(a, c) \in A \times C; \exists b \in B, (a, b) \in R, (b, c) \in S\}$$

• preorder A. elements of $\mathcal{P}$ (preordered set)

→ • relation ($a \rightarrow b = a \leq b$)

Relation has to be

• reflexive

• transitive

• Monoid A. * (one object)

→ • elements of monoid M

$$\bullet id_* = 1$$

$$\bullet \text{composition: } x \circ y = x \cdot y$$

• $\mathcal{C}^{op}$ (dual)

A. objects of $\mathcal{C}$ ($A \mapsto A^*$)

$$\rightarrow \bullet B^* \xrightarrow{f^*} A^* \iff A \xrightarrow{f} B$$

$$\bullet f^* \circ g^* = (g \circ f)^*$$

• $\underline{\mathcal{C}} \rightarrow$ (arrow category)

A. arrows of $\mathcal{C}$

$\rightarrow$ • $g: f \rightarrow f'$ is a pair (g_1, g_2) such that diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{g_1} & A' \\ f \downarrow & & \downarrow f' \\ B & \xrightarrow{g_2} & B' \end{array} \quad (g_2 \circ f = f' \circ g_1)$$

• $\mathcal{C}/C$ (slice category)

A. arrows $A \rightarrow C$

$\rightarrow$ • $a: f \rightarrow g$ (where $f: A \rightarrow C, g: B \rightarrow C$)

is an arrow $a: A \rightarrow B$

such that $g \circ a = f$

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ f \searrow & & \swarrow g \\ & C & \end{array}$$

Construction on categories

• Product $A \times B$

The product of two objects A and B is

1. An object $A \times B$
2. Two projection morphisms $\pi_1: A \times B \rightarrow A$, $\pi_2: A \times B \rightarrow B$, such that $\forall X \in \text{Ob } \mathcal{C}. \forall f: X \rightarrow A. \forall g: X \rightarrow B. \exists \langle f, g \rangle: X \rightarrow A \times B. \pi_1 \circ \langle f, g \rangle = f \wedge \pi_2 \circ \langle f, g \rangle = g$

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\ & \searrow f & \uparrow \exists \langle f, g \rangle & \nearrow g & \\ & X & & & \end{array}$$

Category with finite products has

1. Terminal object
2. Binary product $A \times B$ for every pair A, B

Properties

- uniqueness up to isomorphism
(There exist many products, but they are isomorphic)
- $A \times B \cong B \times A$
- $(A \times B) \times C = A \times (B \times C)$
- $A \times 1 = A$

