

Category

has • objects
• morphisms

satisfies • composition
• associativity
• identity

Terminal object 1

$$\forall A \in \text{Ob } \mathcal{C}, \exists ! \iota_A: A \rightarrow 1.$$

universal property for product

The product of two objects A and B is

1. An object $A \times B$
2. Two projection morphisms $\pi_1: A \times B \rightarrow A$, $\pi_2: A \times B \rightarrow B$,
such that $\forall X \in \text{Ob } \mathcal{C}, \forall f: X \rightarrow A, \forall g: X \rightarrow B$,
 $\exists \langle f, g \rangle: X \rightarrow A \times B$. $\pi_1 \circ \langle f, g \rangle = f \wedge \pi_2 \circ \langle f, g \rangle = g$

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\ & \swarrow f & \uparrow \exists \langle f, g \rangle & \searrow g & \\ & X & & & \end{array}$$

Category with finite products has

1. Terminal object
2. Binary product $A \times B$ for every pair A, B

Properties

- uniqueness up to isomorphism
(There exist many products, but they are isomorphic)
- $A \times B \cong B \times A$
- $(A \times B) \times C = A \times (B \times C)$
- $A \times 1 = A$

A group object

has • an object G

- morphisms
 1. $m: G \times G \longrightarrow G$
 2. $i: G \longrightarrow G$
 3. $e: 1 \longrightarrow G$

satisfying

- associativity $m \circ (m \times \text{id}_G) = m \circ (\text{id}_G \times m)$

$$\begin{array}{ccc}
 G \times G \times G & \xrightarrow{m \times \text{id}_G} & G \times G \\
 \downarrow \text{id}_G \times m & & \downarrow m \\
 G \times G & \xrightarrow{m} & G
 \end{array}$$

- identity law $m \circ (e \times \text{id}_G) = \lambda_G$ $m \circ (\text{id}_G \times e) = \rho_G$

where $\lambda_G: 1 \times G \longrightarrow G$ and $\rho_G: G \times 1 \longrightarrow G$

$$\begin{array}{ccccc}
 1 \times G & \xrightarrow{e \times \text{id}_G} & G \times G & \xleftarrow{\text{id}_G \times e} & G \times 1 \\
 & \searrow \lambda_G & \downarrow m & & \swarrow \rho_G \\
 & & G & &
 \end{array}$$

- inverse law $m \circ (i, \text{id}_G) = e \circ !_G = (i, \text{id}_G) \circ m$

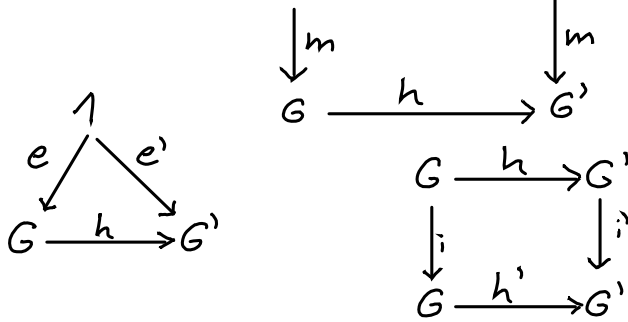
where $\Delta_G = \langle \text{id}_G, \text{id}_G \rangle: G \longrightarrow G \times G$

$$\begin{array}{ccccc}
 & & G & & \\
 & \swarrow (i, \text{id}_G) & \downarrow !_G & \searrow (\text{id}_G, i) & \\
 G \times G & & 1 & & G \times G \\
 & \searrow m & \downarrow e & \swarrow & \\
 & & G & &
 \end{array}$$

Category $\mathbf{Grp}$ has

- objects: Group objects
- morphisms: $(G, m, e, i) \longrightarrow (G', m', e', i')$

such that $G \times G \xrightarrow{h \times h} G' \times G'$



commutes

Algebraic signature

is a tuple (S, Ω, ar)

- $S, \dots$ Sorts: A set of types (or sorts)
- $\Omega, \dots$ Operation symbols: A set of operation names
- $ar, \dots$ Arity function: A mapping that assigns $f \in \Omega$ its input and output type
(if $|S|=1$ then arity represents number of inputs)

Terms X is a set of variables

- variable $x \in X$ is a term
- if $f \in \Sigma$ is an n -ary operation symbol and $t_1, \dots, t_n$ are terms, then $f(t_1, \dots, t_n)$ are terms

Equation over a signature Σ is a pair (t_1, t_2) .

They state, that two different term trees must be treated as equal when evaluated in a model.

Algebraic theory consists of

- signature Σ
 - a set of equations
- $T = (\Sigma, \mathcal{E})$

T-Object in category with finite products is

- object A
- for each n -ary operation $f \in \Sigma$ a morphism $A^n \longrightarrow A$, such that all equations in $\mathcal{E}$ are true in A

Lawvere theory is a category $\mathbb{T}$ with

- objects: $\underline{n} \quad \forall n \in \mathbb{N}$
- morphisms: $\underline{m} \rightarrow \underline{n} : n\text{-tuples}$
 $(t_1(x_1 \dots x_n), \dots, t_m(x_1 \dots x_n))$
- composition: substitution
- identity $\underline{n} \rightarrow \underline{n} : \text{tuple } (x_1 \dots x_n)$
- terminal object: $\underline{m} \times \underline{n} = \underline{m+n}$

$\text{Mod}(\mathbb{T}, \mathbb{C})$... category of $\mathbb{T}$ -objects in $\mathbb{C}$

- objects: $\mathbb{T}$ -objects
- morphisms: $f: A \rightarrow B$ such that square

$$\begin{array}{ccc} A^n & \xrightarrow{f^n} & B^n \\ \downarrow \sigma_A & & \downarrow \sigma_B \\ A & \xrightarrow{f} & B \end{array} \quad \begin{array}{l} \text{commutes for} \\ \text{all } \sigma \text{ of arity } k. \end{array}$$

$\text{FP}[\mathbb{T}, \mathbb{C}]$... category of finite product preserving func.

- objects: functor $F: \mathbb{T} \rightarrow \mathbb{C}$ such that

- $F(\underline{0}) = \text{terminal object in } \mathbb{C}$

- $F(\underline{m+n}) = F(\underline{m}) \times F(\underline{n})$

(The functor is determined by $F(\underline{1}) = A$)

- morphisms: natural transformation

$\alpha: F \Rightarrow G$, so that square

$$\begin{array}{ccc} F(n) & \xrightarrow{\alpha_n} & G(n) \\ \downarrow F(t) & & \downarrow G(t) \\ F(m) & \xrightarrow{\alpha_m} & G(m) \end{array}$$

a family of morphisms

Theorem: $\text{Mod}(\mathbb{T}, \mathbb{C}) \cong \text{FP}[\mathbb{T}, \mathbb{C}]$

Cartesian closed category

- has terminal object
- any two objects X, Y have a product $X \times Y$ in $\mathcal{C}$
- any two objects X, Y have exponential Y^X

Exponential object Y^X is object with a morphism: $\epsilon: Y^X \times X \rightarrow Y$ such that for any morphism $f: Z \times X \rightarrow Y$ there exists a unique morphism $\Lambda_f: Z \rightarrow Y^X$ such that diagram

$$\begin{array}{ccc} Y^X \times X & \xrightarrow{\epsilon} & Y \\ \Lambda_f \times \text{id} \uparrow & & \uparrow f \\ Z \times X & & \end{array}$$

commutes

- exponentials are unique up to isomorphism.

- X is exponentiable if Y^X exists $\forall Y \in |\mathcal{C}|$
- X is exponentiable $\Rightarrow (-)^X$ is a functor
- X is exponentiable $\Leftrightarrow$ the functor $(-) \times X$ has a right adjoint.

Natural transformation η from F to G , where $F, G: \mathcal{C} \rightarrow \mathcal{D}$ are functors, is a family of morphisms $\eta_x: F(x) \rightarrow G(x)$, such that

$$\begin{array}{ccc} F(x) & \xrightarrow{\eta_x} & G(x) \\ \downarrow F(f) & & \downarrow G(f) \\ F(y) & \xrightarrow{\eta_y} & G(y) \end{array} \quad \text{commutes.}$$

Right adjoint of a functor $F: \mathcal{C} \rightarrow \mathcal{D}$

is a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that

$$\forall C \in \mathcal{C}. \operatorname{Hom}_{\mathcal{D}}(F(C), D) \cong \operatorname{Hom}_{\mathcal{C}}(C, G(D)).$$

So that $\operatorname{Hom}_{\mathcal{D}}(F(-), D) \cong \operatorname{Hom}_{\mathcal{C}}(-, G(D))$

is a natural isomorphism

λ -theory $\mathcal{T}(\text{Base}, \Sigma, \mathcal{E})$

- Base ... set of base types
- Σ ... signature with elements $c:A$ such that each element has exactly one type
- $\mathcal{E}$... a set of typed equations $\Gamma \vdash s=t:A$ between terms $\Gamma \vdash s:A$ and $\Gamma \vdash t:A$

Simply typed λ -calculus

- Base ... a set of base types
- types ... $A, B ::= a \in \text{Base} \mid A \times B \mid 1 \mid A \rightarrow B$
- terms ... $\Gamma \vdash t:A$
- context Γ ... $x_1:A_1, \dots, x_n:A_n$, such that $i \neq j \Rightarrow x_i \neq x_j$

Pravila:

- $\frac{x_1:A_1 \dots x_n:A_n \vdash x_i:A_i}{\text{equality}} \quad \frac{\Gamma \vdash t=t':A}{\Gamma \vdash s=s':A} + \text{symetry} + \text{transitivity}$
- $\frac{\Gamma \vdash s=s':A \quad \Gamma \vdash t=t':B}{\Gamma \vdash (s, t) = (s', t') : A \times B} \quad \frac{\Gamma, x:A \vdash t=t':B}{\Gamma \vdash \lambda x. t = \lambda x. t' : A \rightarrow B}$

β -rules

- $\frac{\Gamma \vdash t_1:A_1 \quad \Gamma \vdash t_2:A_2}{\Gamma \vdash \pi_i(t_1, t_2) = t_i : A_i}$
- $\frac{\Gamma, x:A \vdash s:B \quad \Gamma \vdash t:A}{\Gamma \vdash (\lambda x:A. s)t = s[t/x] : B}$

η -rules

- $\frac{\Gamma \vdash u:1}{\Gamma \vdash u = * : 1} \quad \frac{\Gamma \vdash u:A \times B}{\Gamma \vdash u = (\pi_1 u, \pi_2 u) : A \times B}$
- $\frac{\Gamma \vdash u:A \rightarrow B}{\Gamma \vdash u = \lambda x:A. ux : A \rightarrow B}$

T-model in CCC

- objects: a^c for all $a \in \text{Base}$

$$(A \times B)^c = A^c \times B^c$$

- $1^c = 1$

- $(A \rightarrow B)^c = A^c \rightarrow B^c$

- arrows: $\Gamma^c \xrightarrow{\Gamma \vdash t:A} A^c$

where $()^c = 1$

$$(\Gamma, x:A)^c = \Gamma^c \times A^c$$

- subject to condition

$$\Gamma \vdash s = t : A \Rightarrow (\Gamma \vdash s : A)^c = (\Gamma \vdash t : A)^c$$

constants: $1 \rightarrow A^c \quad \forall c:A \in \Sigma$

Theorem:

$$\Gamma \vdash_{\tau} s = t : A \Leftrightarrow \forall C \in \text{CCC}. \forall T \text{ model in } C$$

$$(\Gamma \vdash s : A)^c =$$

$$\forall C \in$$