

Categorical logic

On course webpage:

- notes
- links to videos
- external material

Assessment

- presentations in the seminar slot
- oral exam

Algebraic theories

Perspectives

- Algebraic structure on sets vs. algebraic structure on "spaces"

In what generality can we interpret algebraic structure

- When are two notions of algebra equivalent
- Categorification: presenting mathematical structures thru the lens of category theory

E.g. algebraic theories as Lawvere theories
categories of algebras as functor category
(functional semantics)

A ^{topological} group is a ^{top. space} ~~set~~ G with ^{continuous} $m: G \times G \rightarrow G$, ^{topological product}
 $e \in G$ ($G^0 \rightarrow G$), $i: G \rightarrow G$
 $m(x, m(y, z)) = m(m(x, y), z)$
 $m(e, x) = x = m(x, e)$
 $m(i(x), x) = m(x, i(x)) = e$

More generally we can define a group object in a category $\mathcal{C}$ in any category $\mathcal{C}$ with finite products

A category with ^{final products} has

- binary products: for any ob X, Y we have

$$X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$$

such that for any span $(x \in z \in y)$ there exists $\exists! z \xrightarrow{(p, g)} X \times Y$ such that

$$\begin{array}{ccccc} X & \xleftarrow{\pi_1} & X \times Y & \xrightarrow{\pi_2} & Y \\ & \nwarrow p & \uparrow (p, g) & \nearrow g & \\ & & Z & & \end{array} \quad \text{commutes}$$

- terminal object 1

for any object $X \in \mathcal{C}$ there exists a unique map from $X \rightarrow 1$

A **group object** in a category $\mathcal{C}$ with finite products is an object G plus morphisms

$$m: G \times G \longrightarrow G$$

$$e: 1 \longrightarrow G$$

$$i: G \longrightarrow G$$

satisfying

$$G \times (G \times G) \xrightarrow{G \times m} G \times G$$

$$\cong \downarrow ((\pi_1, \pi_1 \circ \pi_2), \pi_2 \circ \pi_2) \downarrow m$$

$$(G \times G) \times G \xrightarrow{m \times G} G \times G \xrightarrow{m} G$$

$$\begin{array}{ccc} G & \xrightarrow{(e \circ !G, id_G)} & G \\ & \searrow id_e & \downarrow \\ & & G \end{array}$$

+2 for inverse

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \\ Z \times X & \xrightarrow[\cong]{(\pi_1, f \circ \pi_1)} & Z \times Y \end{array}$$

The category $\text{Grp}_{\mathcal{C}}$ of group objects in $\mathcal{C}$ is given by:

- Objects: Group objects in $\mathcal{C}$ (G, m, e, i)
- Morphisms $(G, M, e, i) \xrightarrow{h} (G', M', e', i')$ in $\mathcal{C}$ such that

$$\begin{array}{ccc} G \times G & \xrightarrow{h \times h} & G' \times G' \\ m \downarrow & & \downarrow m \\ G & \xrightarrow{\quad} & G' \end{array}$$

$$\begin{array}{ccc} & 1 & \\ e \swarrow & & \searrow e' \\ G & \xrightarrow{h} & G' \end{array}$$

$$\begin{array}{ccc} G & \xrightarrow{h} & G' \\ i \downarrow & & \downarrow i' \\ G' & \xrightarrow{\quad} & G' \end{array}$$

- identities $(G, m, i, e) \xrightarrow{id_G} (G, m, i, e)$
- composition $(G, m, i, e) \xrightarrow{h} (G', m', i', e')$
 $\searrow h \circ h \quad \downarrow h'$
 $\text{in } \mathcal{C} \searrow (G'', m'', i'', e'')$

An **algebraic signature** Σ is a collective of operations, where each operation f has an associated arity $k \in \mathbb{N}$ ($\mathbb{N}$ includes 0)

The terms T are built by:

- variables x are terms
- if $t_1, \dots, t_k$ are terms and f an operation of arity k , then $f(t_1, \dots, t_k)$ is a term.

Notation $t(x_1, \dots, x_n)$ to mean all variables in t contained in the ~~term~~ set $\{x_1, \dots, x_n\}$

Equation: A pair of terms (t, t') written $t = t' \quad (x_1, \dots, x_n)$

Algebraic theory $T = \Sigma + \mathcal{E}$
 $\uparrow \quad \uparrow$
 signature collection of equations

A T -object in $\mathcal{C}$ (with final product) is:

- an object A of $\mathcal{C}$
- for every operation f in $f \in \Sigma$ of arity k we have a morphism $A^k \xrightarrow{f^A} A$

$$A^0 := 1$$

$$A^1 := A$$

$$A^2 := A \times A$$

$$A^{n+1} = A^n \times A$$

such that all equations in $\mathcal{E}$ are true in A

We interpret any term $t(x_1 \dots x_n)$ as a morphism $A^n \xrightarrow{t^A} A$ where $x_1 \dots x_n$ are distinct

$$A^n \xrightarrow{x_i} A := A^n \xrightarrow{\pi_i} A$$

$$A^n \xrightarrow{t(x_1 \dots x_k)} A := A^n \xrightarrow{(t_1^A \dots t_k^A)} A^k \xrightarrow{t^A} A$$

This gives us an interpretation of all terms

An equation $t = t'$ is satisfied if

$$A^n \xrightarrow{t^A} A = A^n \xrightarrow{(t')^A} A$$

The Category $\text{Mod}(\mathcal{T}, \mathcal{C})$ of $\mathcal{T}$ -objects in $\mathcal{C}$ is defined by

- Objects: $\mathcal{T}$ -objects in $\mathcal{C}$ $(A, (f^A)_{f \in \Sigma})$
- Morphisms $(A, (f^A)_{f \in \Sigma}) \rightarrow (B, (f^B)_{f \in \Sigma})$
morphisms $A \rightarrow B$ s.t. $\forall f \in \Sigma$ of arity k

$$\begin{array}{ccc} A^k & \xrightarrow{h^k} & B^k \\ f^A \downarrow & & \downarrow f^B \\ A & \xrightarrow{h} & B \end{array}$$

The Lawvere theory $\mathbb{T}$ of an algebraic theory $T = (\Sigma, E)$

$\mathbb{T}$ objects: $\underline{n} \quad n \in \mathbb{N}$

morphisms: $\underline{m} \rightarrow \underline{n} \quad \underline{n} = \underline{1}^n$

are n -tuples of terms where $([t_1], \dots, [t_n])$
 $t_1, \dots, t_n \quad (x_1, \dots, x_m)$

$\uparrow$
 equivalence
 classes of terms
 modulo provable
 equality from equations

identity $\underline{n} \rightarrow \underline{n} := \underline{n} \xrightarrow{([x_1], \dots, [x_n])} \underline{n}$

composition $\underline{l} \xrightarrow{\frac{([t_1], \dots, [t_m])}{(x_1, \dots, x_l)}} \underline{m} \xrightarrow{\frac{([t'_1], \dots, [t'_n])}{(x_1, \dots, x_m)}} \underline{n}$

$\underline{l} \xrightarrow{\frac{([t_1][t_1 \dots t_m / x_1 \dots x_m], \dots)}{(x_1, \dots, x_l)}} \underline{n}$

Two algebraic theories T, T' are equivalent if
 corresponding lawvere theories are equivalent

- category
- equivalence of category
- functors
- natural transformations

$\mathbb{T}$ has final product:
 Terminal object $\underline{0}$
 $\underline{m \times n} = \underline{m+n}$

Theorem: $\text{Mod}(\mathbb{T}, \mathbb{C}) \simeq \text{FP}[\mathbb{T}, \mathbb{C}]$
 Final product preserving functors
 objects: $\nearrow$
 arrows: natural transformation

A functor $\mathbb{C} \rightarrow \mathbb{D}$ (categories with final product)
 preserves final product if:
 • $F(1_{\mathbb{C}})$ is terminal object
 • $F(x \times y)$

$\left(\begin{array}{l} F(1_{\mathbb{C}}) = 1_{\mathbb{D}} \\ F(x \times y) = F(x) \times F(y) \end{array} \right) \dots$ preserving product on the nose

$$F x \xleftarrow{F(\pi_2)} F(x \times y) \xrightarrow{F(\pi_1)} F y$$
 is a product diagram in $\mathbb{D}$

Today

- simply typed λ -calculus
interpreted in cartesian closed categories

Cartesian-closed category $\mathcal{C}$

- has terminal product
- any two objects A, B have a product $A \times B$ in $\mathcal{C}$
- any two objects X, Y have exponential Y^X

exponential object has a structure:

- $Y^X \times X \xrightarrow{\epsilon} Y$ "eval"

such that for any map

$$Z \times X \xrightarrow{f} Y \quad \text{there exist a}$$

unique map $Z \xrightarrow{\Lambda_f} Y^X$
satisfying

$$\begin{array}{ccc} Y^X \times X & \xrightarrow{\epsilon} & Y \\ \Lambda_f \times X \swarrow & & \nearrow f \\ Z \times X & & \end{array}$$

Exponentials are unique up to isomorphisms.

$$\text{Suppose } (X \Rightarrow Y) \times X \xrightarrow{\epsilon} Y$$

$$(X \Rightarrow' Y) \times X \xrightarrow{\epsilon'} Y$$

are both exponentials

$$\text{then } (X \Rightarrow' Y) \xrightleftharpoons[\Lambda_\epsilon]{\Lambda_{\epsilon'}} (X \Rightarrow Y)$$

is a pair and is inverse

Examples of cartesian closed categories (ccc)

- Set
- Boolean algebra (More generally Heyting algebras)
- Cat (The category of categories)

An object X is **exponentiable**, if for any $Y \in \mathcal{C}$, the exponential Y^X exists

In $\mathbf{Top}$ a space X is exponentiable $\Leftrightarrow$ it is core compact ($\mathcal{O}(X)$ is a continuous lattice (mreža))

- convenient categories of topological spaces

if X is exponentiable then $(-)^X: \mathcal{C} \rightarrow \mathcal{C}$ is a functor

$$\begin{array}{ccc} Y & \longrightarrow & Y^X \\ f \downarrow & & \downarrow \wedge (f \circ e_{X,Y}) \\ Y' & \longrightarrow & Y'^X \end{array}$$

X is exponentiable $\Leftrightarrow$ the functor $(-)\times X$
 has a right adjoint ($G: \mathcal{C} \rightarrow \mathcal{D}$ functor
 so that for all object C, D
 $\text{hom}_{\mathcal{C}}(FD, C) \cong \text{hom}_{\mathcal{D}}(D, GC)$

Simply-typed λ -calculus

Base : a set of base types

types $A, B, \dots ::= a \in \text{Base} \mid A \times B \mid 1 \mid A \rightarrow B$

terms $\Gamma \vdash t : A$

A context Γ is $x:A_1, \dots, x_k:A_k$ ($k \geq 0$) such
 that $i \neq j \Rightarrow x_i \neq x_j$

$$\frac{}{x_1:A_1, \dots, x_n:A_n \vdash x_i:A_i}$$

Equality relation

$$\frac{\Gamma \vdash t:A}{\Gamma \vdash t=t:A}$$

$$\frac{\Gamma \vdash s=t:A}{\Gamma \vdash t=s:A}$$

+ transitivity

congruence relation

$$\frac{\Gamma \vdash s=s':A \quad \Gamma \vdash t=t':B}{\Gamma \vdash (s, t) = (s', t') : A \times B}$$

$$\frac{\Gamma, x:A \vdash t=t':B}{\Gamma \vdash \lambda x:A. t = \lambda x:A. t' : A \rightarrow B}$$

β -rule

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash \pi_i (t_1, t_2) = t_i : A_i}$$

$$\frac{\Gamma, x:A \vdash s : B \quad \Gamma \vdash t : A}{\Gamma \vdash (\lambda x:A. s) \doteq s[t/x] : B}$$

η -rules

$$\frac{\Gamma \vdash u : 1}{\Gamma \vdash u = * : 1} \qquad \frac{\Gamma \vdash u : A \times B}{\Gamma \vdash u = (\pi_1 u, \pi_2 u) : A \times B}$$

$$\frac{\Gamma \vdash u : A \rightarrow B}{\Gamma \vdash u = \lambda x:A. ux : A \rightarrow B} \quad x \notin 1$$

λ -theory $\mathcal{T}(\text{Base}, \Sigma, \mathcal{E})$

- Base is a set of base types
- Σ a signature with elements $c:A$ such that each c in Σ has exactly one type

Add rule

$$\frac{}{\pi \vdash c:A} \quad c:A \in \Sigma$$

κ constant

- $\mathcal{E}$ is a set of typed equations $\pi \vdash s=t:A$ between terms $\pi \vdash s:A$ $\pi \vdash t:A$

Add rule

$$\frac{}{\pi \vdash s=t:A} \quad \pi \vdash s=t:A \in \mathcal{E}$$

Examples

every algebraic theory gives a λ -theory

Base: $\{a\}$

for every operation of arity k we have

$$c_p: a \times \dots \times a \rightarrow a$$

for every equation ~~we have~~ $t=t'$ with

variables $x_1, \dots, x_k$ $x_i:a, x_1 \dots x_k \vdash t_1=t=t':a$

Example (Gödel's system T)

Base (N)

Signature $0: N$, $\text{succ}: N \rightarrow N$, $\{R_A: A \rightarrow (A \rightarrow A) \rightarrow (N \rightarrow A)\}$ ^{recursor}

Equations $\mathcal{E}$: $x:A \quad f:A \rightarrow A \vdash R_A x f 0 = x$

$x:A \quad f:A; n:N \vdash R_A f (\text{succ } n) = f(R_A x f n): A$

A T -model in a CCC $\mathcal{C}$ is given by

- An in $Q^{\mathcal{C}} \in |\mathcal{C}|$ for every $a \in \text{Base}$
↳ Objekt

This determines interpretation $A^{\mathcal{C}}$ of any type A

$$(A \times B)^{\mathcal{C}} := A^{\mathcal{C}} \times B^{\mathcal{C}}$$

$$1^{\mathcal{C}} := \underline{1}$$

$$(A \rightarrow B)^{\mathcal{C}} := (B^{\mathcal{C}})^{A^{\mathcal{C}}}$$

- A typed constant $c: A \in \Sigma$ as arrows

$$\underline{1} \xrightarrow{c^{\mathcal{A}}} A^{\mathcal{C}}$$

Context interpreted by

$$\emptyset^{\mathcal{C}} := \underline{1}$$

$$\Gamma, x:A := \Gamma^{\mathcal{C}} \times A^{\mathcal{C}}$$

In general a term $\Gamma \vdash t:A$ will be interpreted as a morphism $\Gamma^{\mathcal{C}} \xrightarrow{\Gamma \vdash t:A} A^{\mathcal{C}}$

this involves a lengthy definition

- subject to the condition that every equation $\Gamma \vdash s \equiv t:A$ (every equation provable in T) holds $(\Gamma \vdash s:A)^{\mathcal{C}} = (\Gamma \vdash t:A)^{\mathcal{C}}$

Theorem: Soundness and completeness

Following are equiv.

$$1) \Leftrightarrow \Gamma \vdash_T s = t : A$$

$$2) \Leftrightarrow \forall C \in \mathcal{CCC} \text{ if } T \text{ model in } C$$

$$(\Gamma \vdash s : A)^C = (\Gamma \vdash t : A)^C : \Gamma^C \rightarrow A^C$$

(every derivable equation in the model is provable
and the other way around)

1) $\Rightarrow$ 2) is soundness

2) $\Rightarrow$ 1) is completeness