

KOMA

Izpit: 25.1 (21.2)
4.2

Kalokviji:

- 1) torek 24.11 ob 17.30
- 2) torek 12.1 ob 17.30

1) Naj bo k polje in $k[V]$ $k[X]$ -algebra
 podana s $k[X] \rightarrow k[V]$
 $X \mapsto V^2$
 $\sum \mu_i X^i \mapsto \sum \mu_i V^{2i}$

$k[V]$
 $\uparrow$
 kelobar
 $k[X]$ -modul
 $\lambda(x \cdot y) = (\lambda x) \cdot y = x(\lambda y)$

a) pokaži da je $k[V]$ prosti modul $k[X]$
 in $k[V] \cong \frac{k[X][T]}{(T^2 - X)}$

b) Pokaži $B := k[V] \otimes_{k[V]} k[V] \cong k[V, W] / (V^2 - W^2)$

c) Če $\text{char } k \neq 2$ pokaži da B ni cel kelobar,
 in nima nilpotentov in $B \cong \{ (f, g) \in k[V] \times k[V] : f(0) = g(0) \}$

d) Naj bo $\text{char } k = 2$. Poišči nilpotent (neničeln) v B

$$a) \quad \lambda_0 + \lambda_1 v + \dots + \lambda_n v^n$$

$$x \cdot 1 = v^2$$

$$x \cdot v = v^3$$

$$x \cdot v^n = v^{n+2}$$

$$(1+x) \cdot 1 = 1+v^2$$

$$\{1, v\}$$

$$\lambda_0 + \lambda_1 v + \dots + \lambda_n v^n =$$

$$(\lambda_0 + \lambda_2 v^2 + \dots) + v(\lambda_1 + \lambda_3 v^2 + \dots) =$$

$$= (\lambda_0 + \lambda_2 x + \dots) \cdot 1 + (\lambda_1 + \lambda_3 x + \dots) v$$

$$k[v] \cong \frac{k[x][T]}{(T^2 - x)}$$

$$a_0 + a_1 v \mapsto a_0 + a_1 T$$

Ali to ohranja množenje? ...

2. Način

$$\begin{array}{ccc} k[x][T] & \xrightarrow{\varphi} & k[v] \\ \downarrow \iota & & \nearrow \phi \\ \frac{k[x][T]}{(T^2 - x)} & & \end{array}$$

$$\begin{array}{ccc} \varphi: & T & \mapsto v \\ & x & \mapsto v^2 \end{array}$$

ohranja strukturo modula

$$\begin{aligned} \sum_i \left(\left(\sum_j \mu_{ij} x^j \right) v_i; T^j \right) &\mapsto \left(\sum_i \mu_{ij} v^{2j} \right) \left(\sum_j v_i; v^j \right) \\ \sum_{i,j} \mu_{ij} x^j v_i; T^j &\mapsto \sum_{i,j} \mu_{ij} v_i v^{2j} v^j \quad \checkmark \end{aligned}$$

kač je jedro?

$$ker \varphi = ?$$

$$(T^2 - x) \mapsto v^2 - v^2 = 0 \quad \text{to je v jedru}$$

$$\begin{aligned} \sum v_{jk} x^j T^k &\mapsto \sum v_{jk} x^j v^k = \\ &= p(x) \cdot 1 + q(x) v \\ &= \underbrace{\sum \lambda_i v^{2i}}_{=0} + \underbrace{\sum \gamma_j v^{2j+1}}_{=0} \end{aligned}$$

$$\begin{aligned}
 b) \quad k[V] \otimes \frac{k[x][T]}{(T-x)} &\cong \frac{k[V] \otimes k[x][T]}{(T^2-x)} \cong \\
 &\cong \frac{k[V][T]}{(T^2-x)} \cong \frac{k[V][T]}{(T^2-V^2)}
 \end{aligned}$$

2. način

$$\begin{aligned}
 \varphi: k[V] \otimes k[V] &\longrightarrow \frac{k[V, W]}{(V^2 - W^2)} \\
 \sum \mu_i (p_i(V) \otimes q_i(W)) &\longmapsto \sum \mu_i p_i(V) q_i(W)
 \end{aligned}$$

Univerzalna lastnost & algebre

$$\begin{array}{ccc}
 A & \xrightarrow[\psi]{\text{hom}} & C \\
 B & \xrightarrow[\psi]{\text{hom}} & \\
 & + \text{pogoj komutativnosti} &
 \end{array}
 \rightsquigarrow
 \begin{array}{ccc}
 A \otimes B & \longrightarrow & C \\
 a \otimes b & \longmapsto & \varphi(a) \psi(b)
 \end{array}$$

Preslikevi sta identiteti

$$k[V] \longmapsto \frac{k[V, W]}{(V^2 - W^2)}$$

$$\begin{array}{ccc}
 V & \longmapsto & V \\
 \dots & \dots & \dots \\
 W & \longmapsto & W
 \end{array}$$

To je preslika v eno smer

Druga smer:

$$\begin{aligned}
 \psi: k[V, W] &\longrightarrow k[V] \otimes k[V] \\
 \sum_{ij} \lambda_{ij} V^i W^j &\longmapsto \sum_{ij} \lambda_{ij} (V^i \otimes W^j) \\
 V &\longmapsto (V \otimes 1) \\
 W &\longmapsto (1 \otimes W)
 \end{aligned}$$

Po univerzalni lastnosti je to homomorfizem
 $(V^2 - W^2) \in \ker \psi$

$$(V^2 \otimes 1) - (1 \otimes W^2) = ((V-1) - (V-1))(1 \otimes 1) = 0$$

$\Rightarrow$ to je homomorfizem

$$\frac{k[V, W]}{(V^2 - W^2)} \longrightarrow k[V] \otimes k[V]$$

Ali je to izomorfizem?

$$\phi \circ \bar{\psi} = \text{id}$$

Vstavimo v in w (če je na generatorjih id je povsod id)

$$(\phi \circ \bar{\psi})(v) = \phi(v \otimes 1) = v$$

$$\bar{\psi}(\phi(f \otimes g)) = \psi(f(v) \cdot g(w)) = (f \otimes g)$$

c)

$$B = k[v] \otimes k[w] = \frac{k[v, w]}{(v^2 - w^2)}$$

$$B \cong \{(f, g); h[v] \times h[w]; f(0) = g(0)\}$$

B_cel

- komutativnost ✓
- obrnjivost?
nima delitelja nič



$$\frac{k[v, w]}{(v-w)(v+w)} \longrightarrow$$

crt: I, J ideala, ki sta si tuja v algebri A
 $I + J = A$
 Potem $A/IJ \cong A/I \times A/J$

Ali sta tuja

$$\begin{aligned} I + J &= p(x)(v-w) + g(x)(v+w) = \\ &= (p(x) + g(x))v + (p(x) - g(x))w = (v, w) \end{aligned}$$

Boj splašno

$$A/IJ \cong \{(f, g) \in A/I \times A/J; f \equiv g \pmod{I+J}\}$$

$$A \longrightarrow C$$

$$a \mapsto (a \pmod I, a \pmod J)$$

surjektivnost

$$\frac{I \cap J}{\cong \sqrt{}} = \frac{(v^2 - w^2)}{\cong \sqrt{}}$$

$$\Leftrightarrow f \in I \cap J \Rightarrow (v-w) \mid f \text{ in } (v+w) \mid f$$

ker sta nezvezana in nesociirana

$$\stackrel{\text{UFD}}{\Rightarrow} \text{bo } (v-w)(v+w) \mid f$$

$$\Rightarrow \cong \{(f, g) \in \frac{k[v, w]}{(v-w)} \times \frac{k[v, w]}{(v+w)}\}$$

$$\frac{k[v, w]}{(v \pm w)} \cong k[y]$$

$$f(v, w) \mapsto f(y, \mp y) \quad \text{to je surjektivno}$$

$$\underline{f(0) = g(0)}$$

$$\lambda_0 = \nu_0 \pmod{(v, w)}$$

$$\mathcal{R} = \mathcal{P}(X)$$

crta

$$(2.) A+B = (A-B) \cup (B-A)$$

$\emptyset$

$$A \cdot B = A \cap B$$

$\cup$

a) $I \circ R \Leftrightarrow$ zrite za podmnožice in
 $I \subseteq R$ končne unije

$$I \neq \emptyset$$

$(\Leftarrow)$

$$A, B \in I -$$

$$A+B = \underbrace{(A-B)}_A \cup \underbrace{(B-A)}_B \Rightarrow \checkmark$$

$$A \cdot B = A \cap B \subseteq A$$

$$(\Rightarrow) \text{ Naj } b_0 \quad A \in I \quad B \subseteq A$$

$$\underline{B = A \cap B = A \cdot B \in I}$$

$$A, B \in I$$

$$A \cup B = \underbrace{(A-B) \cap (B-A)}_{\in I = C} \cup \underbrace{(A \cap B)}_{\in I} =$$

$$C \cup (A \cap B) =$$

$$(C - (A \cap B)) \cup ((A \cap B) - C) = A \cap B + C \in I$$

b) Definieren $\langle A \rangle$ in $\langle A, B \rangle$

$$\mathcal{R} \cdot A = \{ B \cap A ; B \in \mathcal{R} \} = \mathcal{P}(A)$$

$$\langle A, B \rangle$$

$$A \cup B \in \langle A, B \rangle \Rightarrow \langle A, B \rangle \subseteq \mathcal{P}(A \cup B)$$

$$\underline{\mathcal{P}(A \cup B) \subseteq \langle A, B \rangle}$$

$$\begin{aligned} c \in \mathcal{P}(A \cup B) &= \underbrace{(C \cap A)}_{\in I} \cup \underbrace{(C \cap B)}_{\in I} = \\ &= ((C \cap A) - (C \cap B)) \cup ((C \cap B) - (C \cap A)) \cup (C \cap A \cap B) \end{aligned}$$

$$\langle A, B \rangle \subseteq \mathcal{P}(A \cup B)$$

$$(C \cap A) + (C \cap B) \subseteq A \cup B \leadsto c \in \mathcal{P}(A \cup B)$$

$$\langle A_1, \dots, A_n \rangle = \langle A_1 \cup A_2, A_3, A_4, \dots, A_n \rangle = \dots = \langle A_1 \cup A_2 \cup A_3, \dots \rangle$$

c) X neskončne

$I = \{ A \subseteq X \text{ je končna} \}$ je ideal kini končno generiran

Recimo da je končno generiran.

Potem $I = \mathcal{P}(A_1 \cup A_2 \cup \dots \cup A_n) = \mathcal{P}(A)$

A je končna	X pa ima neskončno enojce
$\Downarrow$	$\Downarrow$
$\mathcal{P}(A)$ je končan	I je neskončen

Ali je to ideal? Potodi a) ja

Boolev kolesar ... vsi so idempotentni:

$$A \cdot A = A$$

$\forall \mu$ Boolev kolesar $\subseteq \mathcal{P}(S)$ za nekdo končno S
končen

$a \cdot a = a$ ~~$\mathbb{R} = \{a_1, \dots, a_n\}$~~ je boolev

~~Vemo: $ANA = A \Rightarrow R = \mathcal{P}(X)$ je boolev~~

$\mathcal{P}(S)$ ima n elementov

želim $n = 2^k$

$$n = 2^k$$

$$\exists \forall a_i \in A. A \cong \frac{A}{1-a} \times A/a$$

$$\mathbb{R} \cong \mathbb{R} \cdot e \times \mathbb{R} (1-e) \cong$$

Obe sta booleva s končno elementi

indukcija po moči $2^{n_1} \cdot 2^{n_2} = 2^{n_1+n_2}$ ✓

$$n = 2 \dots n = 2^p \quad \checkmark$$

$$\mathbb{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$$

Imamo S s k elementi. $S = [k]$

$\mathcal{P}(S)$

$\mathbb{R} = \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2 \cong S(k)$ je končen boolev
kolesar $f, g, h \in$
izomorfen
 $\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$

$$\mathbb{Z}_2^N$$

$\mathbb{R} = \{a \in \mathbb{Z}_2^N\}$ končno mnogo a : ~~različnih~~ $\{$
ničkončno mnogo $a\}$

$$\mathbb{R} = \bigoplus_N \mathbb{Z}_2$$



Če je neskončen

$\mathcal{P}(S)$ je ali končen ali
neskončen

$$A[[X]] = \{ f = \sum_{n \geq 0} a_n X^n \mid a_n \in A \}$$

$\Rightarrow$ f obrnljiv v $A[[X]]$ $\Leftrightarrow$ a obrnljiv v A
 Me lahko

b) Naj bo K polje

$$I \subseteq K[[X]] \quad I \neq 0 \Rightarrow I = \langle X^n \rangle$$

Naj bo I netrivialni ideal

$$\sum a_n X^n \cdot \sum b_k X^k = \sum_n \sum_{k=0}^n a_n b_k X^n$$

$$\langle X^n \rangle = \{ p(x) X^n \}$$

x je najmanjši n da $a_n \neq 0$ za

$\sum a_k X^k$ lahko zapišemo kot $X^n \sum_{k \geq n} a_{k-n} X^{k-n}$
 $\Rightarrow I \subseteq \langle X^n \rangle$

$$X^n \in I$$

$$\sum_{k=0}^n a_{k+n} X^k$$

$$a_{k+n} \neq 0$$

ker a_n ni 0 je dovolj

c)

$$\text{char } k \neq 2 \quad 1+x = f^2 \quad f \in k[X]$$

$$\left(\sum a_n X^n\right)^2 = \sum_{n=0}^{\infty} \sum_{k=0}^n a_n b_k X^n = 1+X$$

$$(1+X)^{\frac{1}{2}} = \sum a_n X^n \quad \text{Taylorova vrsta}$$

$$= \sum \left(\frac{1}{2}\right)_k X^k$$

$$\left(\frac{1}{2}\right)_k = \frac{\overbrace{\frac{1}{2} \cdot \left(\frac{1}{2}-1\right) \left(\frac{1}{2}-2\right) \dots \left(\frac{1}{2}-k+1\right)}^k}{k!} = 0$$

$$= \frac{2^k}{k!} (-1)(-3)(-5) \dots (1-2k) =$$

$$\frac{(2k-1)!! (-2)^k}{k!}$$

ker pa če $\text{char } k \neq 2$

$$a_0 b_0 = 1$$

$$a_0 b_1 + a_1 b_0 = 1$$

ker je 0 če $a_0 \neq 0$ je to vrsta

$$a_0^2 = 1 \quad \text{Rečimo da } a_0 = 1$$

$$2a_0 a_1 = 1 \quad \Rightarrow a_1 = \frac{1}{2}$$

$$a_1^2 + 2a_2 a_0 = 0 \quad \Rightarrow a_2 = -\frac{1}{2^3}$$

$$2a_3 a_0 + 2a_1 a_2 \Rightarrow a_3 = -\frac{2a_1 a_2}{2a_0} = \frac{1}{2}$$

ker če $\text{char } 2$?

$$x(a_1 a_0 - 2) = 0 \neq 1$$

③ e, e' idempotenti v R

$$(e)^2 = (e)$$

$$(e) = Re$$

$$(e)^2 = (Re)(Re) =$$

$$= \{ae \cdot be\} = \{b e^2\} \subseteq \{abe\} \subseteq (e)$$

$$\underline{(e)} \subseteq \underline{(e)^2}$$

$$\text{obvs. } e = e \cdot e$$

$$\underline{(a)}^2 = (a) \Rightarrow a = (e) \text{ nek } f$$

$$(e) = (e') \Rightarrow e = e'$$

$$Ra = Ra \cdot Ra = \{ba \cdot ca\} = \{bca^2\}$$

$$(a)^2 = (a^2)$$

$$(a) = (a^2)$$

$$Ra = Ra^2$$

$$ba = ca^2$$

$$ka^2 = a \quad /k$$

$$(ka)^2 = ka \Rightarrow ka \text{ je idempotent}$$

$$(ka)R = (R \cdot k) \cdot a \subseteq R \cdot a$$

$$a = ka^2 \Rightarrow a \in R \cdot k \cdot a \Rightarrow Ra \subseteq R \cdot ka$$

Recimo da

$$(e) = (e')$$

$$Re = Re'$$

$$e' = ke$$

$$e = ce'$$

$$e = cke$$