

Hilbertov prostor

X vektorski prostor nad $\mathbb{R}$ (nad $\mathbb{C}$)
skalarni produkt $\langle \cdot \rangle$

$$\langle \cdot \rangle : X \times X \longrightarrow \mathbb{R}$$

$$1) \forall x \in X : \langle x, x \rangle \geq 0$$

$$2) \langle x, x \rangle = 0 \iff x = 0$$

pozitivna
definitnost

$$3) \forall x, y \in X.$$

$$\langle x, y \rangle = \langle y, x \rangle \text{ nad } \mathbb{R}$$

$$\langle x, y \rangle = \overline{\langle y, x \rangle} \text{ nad } \mathbb{C}$$

simetričnost oz antisimetričnost

$$4) \forall x, y, z \in X, \forall \lambda, \mu \in \mathbb{R}$$

$$\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$$

linearnost v prvem faktorju

za sklanj: produktov

Cavaty-schwarzova neenakost

$$\forall x, y \in X$$

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \cdot \sqrt{\langle y, y \rangle}$$

$$\|x\| = \sqrt{\langle x, x \rangle}$$

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Dokaz nad $\mathbb{R}$

$$t \mapsto \langle x + ty, x + ty \rangle = s(t) \geq 0$$

$$= \langle x, x \rangle + 2t \langle x, y \rangle + t^2 \langle y, y \rangle$$

$$= \|x\|^2 + 2t \langle x, y \rangle + t^2 \|y\|^2 \geq 0$$

$$D \leq 0$$

$$D^2 = \langle x, y \rangle^2 - \|x\|^2 \|y\|^2$$

$$\Rightarrow \langle x, y \rangle^2 \leq \|x\|^2 \|y\|^2$$

$$\langle x, y \rangle \leq \|x\| \|y\|$$

enako velja če sta x in y
natanko tedaj

linearno odvisna

Need $\mathbb{C}$:

x, y

$$\exists \alpha \quad |\alpha| = 1$$

$$\langle x, y \rangle = \alpha |\langle x, y \rangle|$$

$$\Rightarrow \langle x, \alpha y \rangle = |\langle x, y \rangle|$$

$$f(t) = \langle x + t\alpha y, x + t\alpha y \rangle =$$

$$= \|x\|^2 + t \langle \alpha y, x \rangle + \underbrace{\langle x, \alpha y \rangle}_{2 \cdot |\langle x, y \rangle|} + t^2 \langle \alpha y, \alpha y \rangle$$

$$= \|x\|^2 + 2t |\langle x, y \rangle| + t^2 \|y\|^2 \geq 0$$

$$D = 4 |\langle x, y \rangle|^2 - 4 \|x\|^2 \|y\|^2 \leq 0$$

Norma nad vektorskim prostorom X

$$\| \cdot \|: X \rightarrow \mathbb{R}$$

$$1) \forall x \in X. \|x\| \geq 0$$

$$2) \|x\| = 0 \Leftrightarrow x = 0$$

$$3) \forall \lambda \in \mathbb{R} \vee \mathbb{C}, \forall x \in X$$

$$\|\lambda x\| = |\lambda| \|x\| \quad \text{homogenost}$$

$$4) \text{trikotniška neenakost}$$

$$\forall x, y \in X. \|x+y\| \leq \|x\| + \|y\|$$

če je $(X, \langle \cdot, \cdot \rangle)$ vektorski prostor
s skalarnim produktom je

$$\|x\| = \sqrt{\langle x, x \rangle} \quad X \text{ vektorski prostor}$$

≥ 0

$$\|x\| = \sqrt{\langle x, x \rangle}$$

1), 2) Skali iz 1) 2) Skalarnege produkta

$$\begin{aligned} 3) \quad \|\lambda x\| &= \sqrt{\langle \lambda x, \lambda x \rangle} = \sqrt{|\lambda|^2 \langle x, x \rangle} = \\ &= |\lambda| \|x\| \end{aligned}$$

4)

$$\|x+y\|^2 = \langle x+y, x+y \rangle =$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle =$$

$$= \|x\|^2 + 2\operatorname{Re}\langle x, y \rangle + \|y\|^2 \leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

$$(X, \langle \cdot, \cdot \rangle) \leadsto (X, \| \cdot \|)$$

$\leadsto$ metric; prostar (X, d)

$$\forall x, y \in X. d(x, y) = \|x - y\|$$

$$1) d(x, y) \geq 0$$

$$2) d(x, y) = 0 \Leftrightarrow x = y$$

$$3) d(x, y) = d(y, x)$$

$$4) \forall x, y, z \in X$$

$$d(x, y) + d(y, z) \geq d(x, z)$$

Def: Hilbertov prostor je vektorski prostor s skalarnim produktom ki je v metriki porjeden: iz skalarnega produkta poln metrični prostor

Opomba: Banachov prostor je vektorski prostor X z normo $\| \cdot \|$ ki je v metriki porjeden: iz norme poln metrični prostor

Zagled:

$$1) \mathbb{R}^n, \quad x = (x_1 \dots x_n) \\ y = (y_1 \dots y_n)$$

$$x \cdot y = x_1 y_1 + \dots + x_n y_n$$

$$\|x\| = \sqrt{x_1^2 + \dots + x_n^2} \quad d_2 \text{ metrika}$$

$$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} \quad \text{je}$$

pači metrični prostor

$(\mathbb{R}^n, \cdot)$ je Hilbertov prostor

$$(\mathbb{R}^n, \|\cdot\|_\infty) \max \{|x_1|, \dots, |x_n|\}$$

norma

$$(\mathbb{R}^n, \|\cdot\|_1) \quad |x_1| + \dots + |x_n|$$

~~Banachova prostora~~
~~ker sta topološko enake $(\mathbb{R}^n, d_2)$~~

~~$\mathbb{R}^n$~~ sta Banachova ker sta
 topološki ekvivalentna $(\mathbb{R}^2, d_2)$

ampak ne pridejo iz skalarnega
 produkta

$$2) \mathbb{C}^n \quad z = z_1, \dots, z_n$$

$$w = w_1, \dots, w_n$$

$$z \cdot w = z_1 \overline{w_1} + \dots + z_n \overline{w_n}$$

$$\|z\| = \sqrt{z_1^2 + \dots + z_n^2}$$

$$d_2(z, w) = \sqrt{|z_1 - w_1|^2 + \dots + |z_n - w_n|^2}$$

$(\mathbb{C}^n, \cdot)$ je Hilbertov prostor

Zafed

$$[a, b] \subseteq \mathbb{R} \quad a < b$$

$$X = C([a, b])$$

$$\langle f, g \rangle = \int_a^b f(x) g(x) dx \quad \text{nad } \mathbb{R}$$

$$3) \langle f, g \rangle = \langle g, f \rangle$$

$$4) \langle \lambda f + \mu g, h \rangle = \lambda \langle f, h \rangle + \mu \langle g, h \rangle$$

$$1) \langle f, f \rangle = \int_a^b f^2(x) dx \geq 0$$

$$2) f = 0 \Rightarrow \langle f, f \rangle = 0$$

$$\int_a^b f^2(x) dx = 0 \Rightarrow f^2 = 0 \text{ skoro všude}$$

$$\Rightarrow f = 0 \text{ povsod kromě znenost}$$

$$\Rightarrow f = 0 \quad (\text{Následující stran})$$

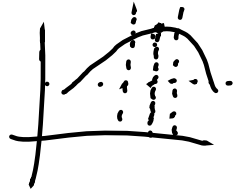
Pokračovat?

$$\int_a^b f^2(x) dx = 0$$

$$\Leftrightarrow \exists x_0 \in [a, b] \text{ s.t. } f(x_0) = 0 \Rightarrow$$

$$\exists \delta > 0 \text{ s.t. } \forall x \in (x_0 - \delta, x_0 + \delta) \cap [a, b].$$

$$|f(x)| \geq \frac{|f(x_0)|}{2}$$



$$\int_a^b f^2(x) dx \geq \int_{(x_0 - \delta, x_0 + \delta) \cap [a, b]} f^2(x) dx \geq \frac{f(x_0)^2}{2} \delta$$

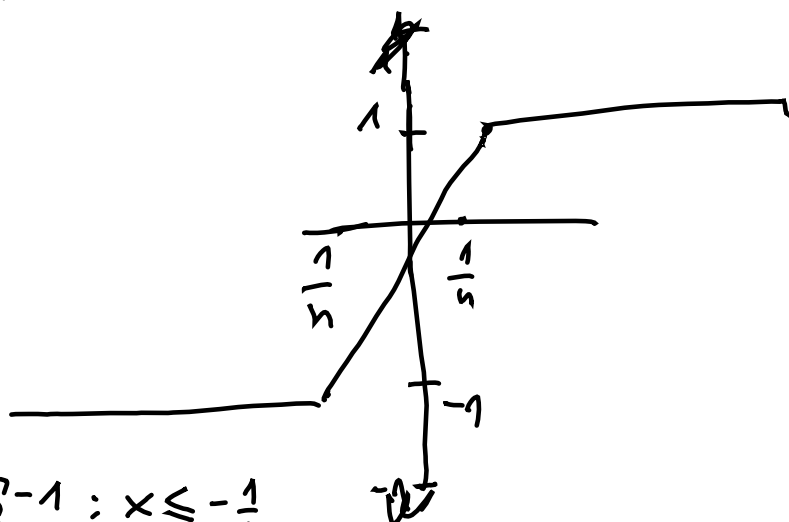
$$\Leftrightarrow f = 0 \text{ a.e.} \Rightarrow f = 0 \text{ a.s.}$$

Trditelj: $(C[a,b], \langle \cdot, \cdot \rangle)$

n: Hilbertov

Dokaz

$f_n(x)$



$$f_n(x) = \begin{cases} -1 & ; x \leq -\frac{1}{n} \\ nx & ; -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 1 & ; x \geq \frac{1}{n} \end{cases}$$

$(f_n)_n$ je Cauchyjeva zaporedje v $(C[-1,1], d)$
 d -metrike iz skalarnega produkta

$$d(f, g)^2 = \int_{-1}^1 |f(x) - g(x)|^2 dx$$

$m > n$

$$d(f_n, f_m)^2 =$$

$$\begin{aligned} & \int_{-\frac{1}{n}}^{\frac{1}{n}} (f_n(x) - f_m(x))^2 dx = \\ & \int_{-\frac{1}{n}}^{\frac{1}{n}} (f_m(x) - f_n(x))^2 dx \leq \int_{-\frac{1}{n}}^{\frac{1}{n}} 1 dx = \frac{2}{n} \end{aligned}$$

$$d(f_n, f_m) \leq \sqrt{\frac{2}{n}} < \varepsilon$$

$(f_n)_n$ je Cauchyjeva v $(C[-1,1], d)$

$\lim f_n$

če obstaja $\lim f_n$ $(f_n)_n \subset C[-1,1]$

nam vspeti, da je $f(x) = \begin{cases} 1 & ; 0 \leq x \leq 1 \\ -1 & ; -1 \leq x < 0 \end{cases}$

$\Rightarrow f_n$ zveza v 0

X

Zagled

$$M = (0, 1)$$

$$d_2(x, y) = |x - y|$$

~~M~~ M n: poln

$M \cup \{0, 1\}$ je poln

Dodati smo mozne limite

Nepolnitet metričnega prostora

(M, d) nepolnimo lahke nepolnemo

$(\bar{M}, \bar{d})$ poln

1) $M \subseteq \bar{M}$

2) $\bar{d}_{M \times M} = d$

3) M je gost v $\bar{M}$

$$L^1(A) = \{f: A \rightarrow \mathbb{R} : \int_A |f| dx < \infty\}$$

5. Kwadratom integrabilna funkcije:

$$L^2([a, b]) = \{f: [a, b] \rightarrow \mathbb{R} : \int_a^b |f|^2 dx < \infty\}$$

Zapad:

$$1) C([a, b]) \subseteq L^2([a, b])$$

$$2) \text{ odseki na zvezne } \subseteq L^2$$

$$3) f(x) = \frac{1}{\sqrt{x-a}} \in L^2$$

$$4) g(x) = \frac{1}{2\sqrt{x-a}} \notin L^2([a, b])$$

Velj: $f \cdot g$

$$|f \cdot g| \leq \frac{|f|^2 + |g|^2}{2} \Rightarrow f \cdot g \in L^1([a, b])$$

$$\langle f, g \rangle = \int_a^b f(x) g(x) dx$$

L^2 je vektorski prostor

$$(f+g)^2 = f^2 + 2fg + g^2$$

$\in L^2 \quad \in L^1 \quad \in L^2$

$$\int_a^b (f+g)^2 dx = \int_a^b f^2 dx + 2 \int_a^b f(x)g(x) dx + \int_a^b g^2 dx < \infty$$

$$\Rightarrow \|f+g\| \in L^2$$

$$C[a,b] \subseteq L^2([a,b])$$

hilbertov

Opomba

$$\forall f \in L^2([a,b]) . \exists f_n \in C[a,b].$$

$$\lim_{n \rightarrow \infty} f_n = f \iff \lim_{n \rightarrow \infty} \|f_n - f\| = 0$$

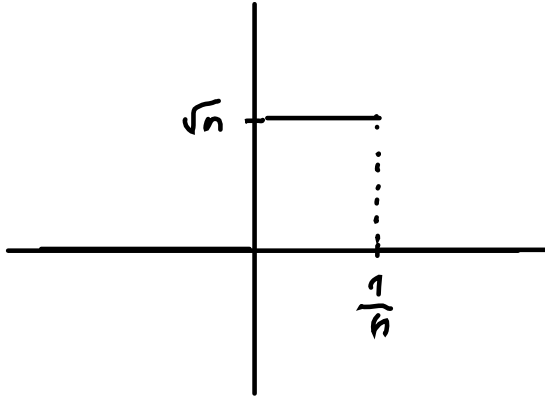
$$\lim_{n \rightarrow \infty} \int_a^b (f_n(x) - f(x))^2 dx = 0$$

Opomba: ved $C: f = u + i v \quad u, v: [a,b] \rightarrow \mathbb{R}$

$$\int_a^b f(x) dx = \int_a^b u(x) dx + i \int_a^b v(x) dx$$

Zgled: na $[0, 1]$

$$f_n: \begin{cases} \sqrt{n} & ; 0 < x \leq \frac{1}{n} \\ 0 & ; \text{sicer} \end{cases}$$

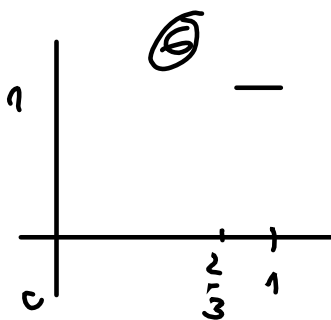
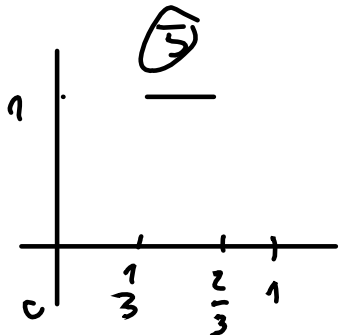
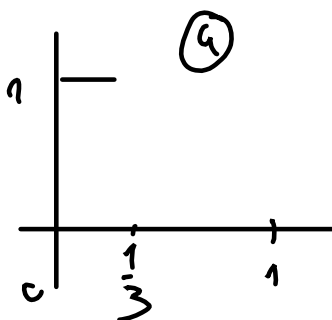
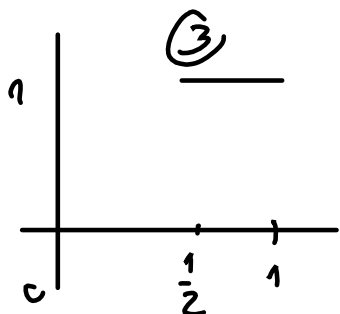
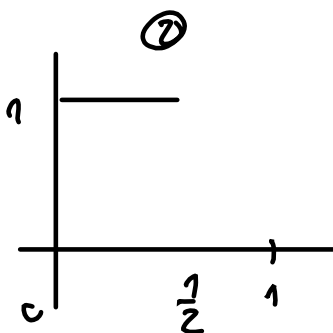
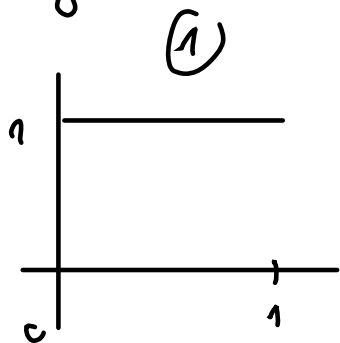


$$\lim f_n(x) = 0 \text{ za } x \in [0, 1]$$

Ali: konvergira v L^2 ?

$$\|f_n - 0\| = \sqrt{\int_0^1 f_n^2(x) dx} = \sqrt{\int_0^{\frac{1}{n}} \sqrt{n}^2 dx} = 1$$

Zgled:



tol

$\lim_{n \rightarrow \infty} f_n(x)$ ne obstaja

$\lim_{n \rightarrow \infty} f_n(x) \in L^2$?

$$\int_0^1 f_n^2(x) dx \leq \frac{1}{n^2} \quad \text{torej limita je } 0$$

Naj bo $(X, \langle \cdot \rangle)$ vek. przestrzeń z skalarnym produktem.



$$A \subseteq X \quad (A \neq \emptyset)$$

$$x, y \in X. \quad x \perp y \Leftrightarrow \langle x, y \rangle = 0$$

(symetryczna relacja)

$$A \subseteq X \quad A^\perp = \{x \in X. x \perp a \text{ dla } \forall a \in A\}$$

ortogonalni: komplement

Trditev: $A^\perp$ je vektorski podprostor X

Dokaz:

$$x \in A^\perp \quad \lambda \in \mathbb{R} \text{ (ali } \mathbb{C})$$

$$a \in A. \quad \langle \lambda x, a \rangle = \lambda \langle x, a \rangle = \lambda \cdot 0 = 0$$

$$x, y \in A^\perp$$

$$\langle x+y, a \rangle = \langle x, a \rangle + \langle y, a \rangle = 0 + 0 = 0$$

Velja: $A \subseteq (A^\perp)^\perp$

Trditev: $v \in X$

$$f(x) = \langle x, v \rangle \quad f: X \rightarrow \mathbb{R}$$

f je zvezna na X

Dokaz:

$$x_1, x_2 \in X$$

$$|f(x_1) - f(x_2)| = |\langle x_1 - x_2, v \rangle| \leq \|x_1 - x_2\| \|v\|$$

f je enakomerno zvezna
(kako Lipschitzova zvezna)

Posledica: $A^\perp$ je zaprt vektorski
podprostor

$C[a,b] \subseteq L^2[a,b]$ ni zaprt podprostor

Dokaz:

zaporedje x_n

$$\lim x_n = x_0 \in X \stackrel{?}{\Rightarrow} x_0 \in A^\perp$$

$$\forall a \in A. \langle x_n, a \rangle = 0 \quad \forall n$$

$$\lim \langle x_n, a \rangle = \langle \lim x_n, a \rangle = \langle x_0, a \rangle = 0$$

$\parallel$
 a

Opomba: $(X, \langle \rangle)$ Hilbertov

$$A \subseteq X \text{ zaprt podprostor} \Rightarrow A = (A^\perp)^\perp$$

Trditev: (Pitagorov izrek)

$(X, \langle \cdot \rangle)$ vektorski prostor s skalarnim produktom

$$x_1, \dots, x_n \in X \quad \forall i \neq j \in [n], x_j \perp x_i \\ (\langle x_j, x_i \rangle = 0)$$

$$\text{Torej } \|x_1\|^2 + \dots + \|x_n\|^2 = \|x_1 + \dots + x_n\|^2$$

Dokaz:

$$\|x_1 + \dots + x_n\|^2 = \langle x_1 + x_2 + \dots + x_n, x_1 + \dots + x_n \rangle$$

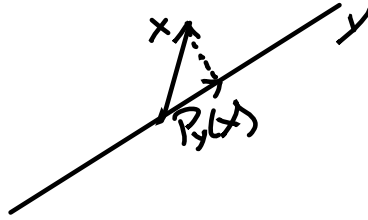
$$= \left\langle \sum_i x_i, \sum_j x_j \right\rangle = \sum_i \sum_j \langle x_i, x_j \rangle =$$

$$= \|x_1\|^2 + \dots + \|x_n\|^2$$

$$(X, \langle \rangle)$$

$$Y \leq X$$

$$x \in X$$



Definicija: Pravokotna projekcija
vektorja x na podprostor Y (če obstaja)
je tak vektor $P_Y(x) \in Y$ da je
 $x - P_Y(x) \in Y^\perp$

Trditev: Če pravokotna projekcija
 x na Y obstaja, je enolično določena
in $P_Y(x)$ je najboljša aproksimacija
vektorja x z vektorji iz Y
(razdalja je najmanjša)
 $(\|x - P_Y(x)\| = \min_{w \in Y} \|x - w\|)$

Dokaz:

$$Y \subseteq X \quad x \in X$$

Domimo da y_1, y_2 sta pravokotni projekciji
 x na Y

$$x - y_1, x - y_2 \in Y^\perp$$

$$(x - y_1) - (x - y_2) \in Y^\perp \quad \swarrow \text{ker vek. prostor}$$

$$y_2 - y_1 \in Y^\perp$$

$$y_2 - y_1 \in Y$$

$$\langle y_2 - y_1, y_2 - y_1 \rangle = 0 \iff$$

$$y_2 - y_1 = 0 \quad y_2 = y_1$$

$$w \in Y$$

$$x - w = \underbrace{x - P_Y(x)}_{\in Y^\perp} + \underbrace{P_Y(x) - w}_{\in Y}$$

Pitagorin izrek:

$$\|x - w\|^2 = \|x - P_Y(x)\|^2 + \|P_Y(x) - w\|^2 \geq \|x - P_Y(x)\|^2$$

Zagled:

$$Y = C[a, b] \quad X = L^2[a, b]$$

$f \in X - Y$ nima pravokotne projekcije

Tak f nima najboljše aproksimacije
z veznimi funkcijami

Komentar: hilbertov + zaprt p-ten ima projekcije

Opombe:

$$1) P_Y^2 = P_Y$$

$$2) x = \underbrace{x - P_Y(x)}_{\in Y^\perp} + \underbrace{P_Y(x)}_{\in Y}$$

$$\|x\|^2 = \|x - P_Y(x)\|^2 + \|P_Y(x)\|^2$$
$$\|x\| \geq \|P_Y(x)\|$$

3) če je P_Y definirana na celotni X ,
potem je P_Y linearna in ortogonalna

$$\|P_{Y_1} - P_{Y_2}\| = \|P_Y(x_1 - x_2)\| \leq \|x_1 - x_2\|$$

P_Y je ortogonalna
linearnost:

$$\lambda x - P_Y(\lambda x)$$

Poglejmo:

$$\lambda x - \lambda P_Y(x) = \underbrace{\lambda(x - P_Y(x))}_{\in Y^\perp}$$

$$\Rightarrow \lambda P_Y(x) \in Y$$

$$\lambda x - \lambda P_Y(x) \in Y^\perp$$

če to je pravokotna projekcija
na X

$$x_1, x_2 \in X$$

$$P_Y(x_1 + x_2) = P_Y(x_1) + P_Y(x_2)$$

$$(x_1 + x_2) - \underbrace{(P_Y(x_1) + P_Y(x_2))}_{\in Y} = x_1 - P_Y(x_1) + x_2 - P_Y(x_2) \in Y^\perp$$

$$\Rightarrow P_Y(x_1 + x_2) = P_Y(x_1) + P_Y(x_2)$$

zato: enolična projekcija

če P_Y definirana

Če je P_Y definiran na X je
Y apit prostor

Dokazi:

$$\{y_j\} \subseteq Y \quad \lim_{j \rightarrow \infty} y_j = y_0$$

$$P_Y(y_j) = y_j$$

$$\lim_{j \rightarrow \infty} y_j = \lim_{j \rightarrow \infty} P_Y(y_j) \stackrel{\text{zveznost}}{=} P_Y(y_0)$$

||
 y_0

Če ima x pravokotno projekcijo na Y
ima tudi pravokotno projekcijo na $Y^\perp$

$$x; P_Y(x)$$

$$P_{Y^\perp}(x) = x - P_Y(x) \in Y^\perp$$

$$\underline{x - (x - P_Y) \in (Y^\perp)^\perp}$$

$$\parallel \\ P_Y(x) \in Y \subseteq (Y^\perp)^\perp$$

Trditveni: Naj bo $Y \leq X$ končno dimenzijski
 podprostor z ortonormirano bazo
 $e_1, \dots, e_n$. $\langle e_i, e_j \rangle = \delta_{ij}$

Naj bo $x \in X$. Tedaj

$$P_Y(x) = \sum_{i=1}^n \langle x, e_i \rangle e_i$$



Opomba: Vsak končno dimenzijski
 podprostor ima pravokotno projekcijo
 definirano na X

In tudi vsi tisti končno kodimenzijski

Dokaz: $P_Y(x) = \sum_{i=1}^n \langle x, e_i \rangle e_i \in Y$

Poglejmo se

$$x - \sum_{j=1}^n \langle x, e_j \rangle e_j \in Y^\perp$$

$$\langle x - \sum_{j=1}^n \langle x, e_j \rangle e_j, e_i \rangle =$$

$$\langle x, e_i \rangle - \sum_{j=1}^n \langle x, e_j \rangle \langle e_j, e_i \rangle =$$

$$\langle x, e_i \rangle - \langle x, e_i \rangle = 0$$

$$(x, \langle \rangle)$$

Sistem vektorjev

$$(e_j)_{j=1}^{\infty}$$

je ortogonalen sistem (OS), če

$$\forall i \neq j. \langle e_i, e_j \rangle = 0$$

Tak sistem je ortonormiran (ONS)

$$\forall i, j. \langle e_i, e_j \rangle = \delta_{ij}$$

Trditev: $(X, \langle \cdot, \cdot \rangle)$ Naj bo $(e_j)_j$ ONS

Naj bo $x \in X$. Tedaj je $\sum |\langle x, e_j \rangle|^2 \leq \|x\|^2$

↑
(Besselova neenakost)

Opomba: $\langle x, e_j \rangle$ so Fourierovi koeficienti
x po ONS $(e_j)_j$

Posledica:

$$\lim_{j \rightarrow \infty} \langle x, e_j \rangle = 0$$

Dokaz: $\gamma_n = \mathcal{L}(\{e_1, \dots, e_n\})$

$$x \in X$$

$$\exists P_\gamma(x) = \sum_1^n \langle x, e_j \rangle e_j$$

je veljina krajša od x
prideleje se

$$\|P_\gamma(x)\|^2 = \sum_1^n \|\langle x, e_j \rangle e_j\|^2 = \sum_1^n |\langle x, e_j \rangle|^2 \leq \|x\|^2$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sum_1^n \|\langle x, e_j \rangle\|^2 = \sum_1^\infty |\langle x, e_j \rangle|^2 \leq \|x\|^2$$

Trditelj: Naj bo $(c_j)_{j=1}^{\infty}$ zaporedje
števil (ali $\mathbb{R}$ ali $\mathbb{C}$) za katero velja
$$\sum_1^{\infty} |c_j|^2 < \infty$$

Naj bo $(X, \langle \cdot, \cdot \rangle)$ Hilbertov prostor
 $(e_j)_{j=1}^{\infty}$ ONS

Torej $\exists x \in X$, za katerega velja $c_j = \langle x, e_j \rangle$
za $\forall j$.

$$x = \sum_1^{\infty} c_j e_j = \lim \sum_1^N c_j e_j$$

$(x, \langle \cdot \rangle)$ h: ortonormal e_j on $\mathcal{H}$

$$x \in \mathcal{H} \sim (\langle x, e_j \rangle)_j \quad \sum_1^\infty |\langle x, e_j \rangle|^2 \leq \|x\|^2$$

$$\Rightarrow \exists \tilde{x} = \sum_1^\infty \langle x, e_j \rangle e_j$$

$$\text{A1: } j \in \mathbb{N} \quad \tilde{x} = x$$

✓

Zemuda

20.2

Zagled (pred prejšnjim delom)

Modelni hilbertov prostor l^2

Prostor zaporedij

$$l^2 = \{ (a_j)_j ; a_j \in \mathbb{R}, \sum_1^\infty |a_j|^2 < \infty \}$$

$$\langle (a_j), (b_j) \rangle = \sum a_j b_j \text{ nad } \mathbb{R}$$

$$\sum a_j \bar{b}_j \text{ nad } \mathbb{C}$$

$$\|a_j\|^2 = \sum |a_j|^2$$

$(X, \langle \rangle)$ hilbertov $(\mathcal{E})$ kans

$$x \mapsto (\langle x, e_j \rangle)_j \in l^2$$

$$e_j = (0, \dots, 0, 1, 0, \dots, 0)$$

Dodaj:

$$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$$

$$1 \Rightarrow 6 \Rightarrow 5$$

$$1) \Rightarrow 2)$$

$$x, y \in X \quad x = \sum_1^{\infty} \langle x, e_j \rangle e_j$$

$$\langle x, y \rangle = \langle \sum_1^{\infty} \langle x, e_j \rangle e_j, y \rangle =$$

$$= \sum_1^{\infty} \langle x, e_j \rangle \langle e_j, y \rangle$$

$$2) \Rightarrow 3)$$

$$x=y \quad \langle x, y \rangle = \|x\|^2 = \sum_1^{\infty} |\langle x, e_j \rangle|^2$$

$$3) \Rightarrow 4)$$

ONS
čeb: b: vsebovan v strogo večjem ONS
pateri $\exists e_0 \perp e_j \quad \forall j \quad \|e_0\|=1$

Vstavimo v parčevalno enačbo

$$\|e_0\|^2 = 1^2 = \sum_1^{\infty} |\langle e_0, e_j \rangle|^2 = \sum_1^{\infty} 0 = 0$$

$$4) \Rightarrow 5)$$

$$x \perp e_j \quad \forall j$$

$$\text{če } x \neq 0 \text{ tvorimo } e_0 = \frac{x}{\|x\|}$$

$(e_0, e_j)_j$ je strogo večji KONS, ker ne more biti *

$$5) \Rightarrow 1)$$

$$x \in X \quad \langle x, e_j \rangle_j$$

$$\tilde{x} = \sum_1^{\infty} \langle x, e_j \rangle e_j \quad \text{ali sta enaka?}$$

$$\tilde{v} = x - \tilde{x} = x - \sum_1^{\infty} \langle x, e_j \rangle e_j$$

$$\langle v, e_i \rangle = \langle x, e_i \rangle - \sum_1^{\infty} \langle x, e_j \rangle \langle e_j, e_i \rangle =$$

$$\langle x, e_i \rangle - \langle x, e_i \rangle = 0$$

$$\tilde{v} = 0 \Rightarrow x = \tilde{x}$$

$$1) \Rightarrow 6) \quad \forall x = \sum_1^{\infty} \langle x, e_j \rangle e_j = \lim_{N \rightarrow \infty} \sum_1^N \langle x, e_j \rangle e_j$$

$$\forall \varepsilon > 0 \quad \exists N_0 \in \mathbb{N} \quad \|x - \sum_1^{N_0} \langle x, e_j \rangle e_j\| < \varepsilon$$

$$6) \Rightarrow 5)$$

$$\text{Naj bo } x \perp e_j \quad \forall j$$

$$\underline{x=0}$$

$$\varepsilon > 0$$

$$\exists \text{ končna linearna kombinacija } \sum_1^N \lambda_j e_j$$

$$\text{da je } \|x - \sum_1^N \lambda_j e_j\| < \varepsilon$$

$$\|x\|^2 = \langle x, x \rangle = \langle x - \sum_1^N \lambda_j e_j, x \rangle \leq$$

↑
ker pravokotnost

$$\leq \|x - \sum_1^N \lambda_j e_j\| \cdot \|x\| < \varepsilon \|x\|$$

$$1) \quad x=0 \Rightarrow \checkmark$$

$$2) \quad x \neq 0 \Rightarrow \|x\| < \varepsilon \quad \forall \varepsilon > 0 \Rightarrow x=0 \quad *$$

Ortodogonalna se na $L^2(-\pi, \pi)$

$$\text{ONS: } \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{\pi}} \sin 2x, \dots$$
$$\dots, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx), \dots$$

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) dx$$

$$\|f\|_2 = \sqrt{\int_{-\pi}^{\pi} f^2(x) dx}$$

$$f_0(x) = \frac{1}{\sqrt{2\pi}} \quad \|f_0\|_2 = \sqrt{\int_{-\pi}^{\pi} \frac{1}{2\pi} dx} = 1$$

$$\sqrt{\int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \sin^2(nx) dx} = \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1 - \cos(2nx)}{2} dx} = 1$$

Opomba: nad $\mathbb{R}$: $\frac{1}{\sqrt{2\pi}} e^{inx}$ $n \in \mathbb{Z}$ ~~vsaj~~ v L^2 ortogonalna
na ONS

$n \neq m$

$$\langle f_n, f_m \rangle = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} e^{inx} \frac{1}{\sqrt{2\pi}} e^{imx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)x} dx$$
$$= \frac{1}{2\pi(n-m)} e^{i(n-m)x} \Big|_{-\pi}^{\pi} = 0$$

$f: (-\pi, \pi) \rightarrow \mathbb{R}$ periodična funkcija
s periodom 2π

Klasični Fourierovi koeficienti

$$f \in L^2(-\pi, \pi)$$

(Recimo da imamo karos)

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad n \in \{0, 1, \dots\}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \langle f, \frac{1}{\pi} \cos(nx) \rangle =$$

$$= \frac{1}{\sqrt{\pi}} \langle f, \frac{1}{\sqrt{\pi}} \cos(nx) \rangle = \sqrt{\pi} a_n$$

$$\langle f, \frac{1}{\pi} \sin(nx) \rangle = \sqrt{\pi} b_n$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot 1 dx = \langle f, \frac{1}{\pi} \rangle = \frac{\sqrt{2}}{\sqrt{\pi}} \langle f, \frac{1}{\sqrt{2\pi}} \rangle =$$

$$\langle f, \frac{1}{\sqrt{2\pi}} \rangle = \frac{\sqrt{\pi}}{\sqrt{2}} a_0$$

če je $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \dots \right\}$ kons
poten velja Parsevalova enakost

$$\|f\|^2 = \int_{-\pi}^{\pi} |f|^2(x) dx = \frac{\pi}{2} a_0^2 + \sum \pi (a_n^2 + b_n^2)$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2$$

Posledica:

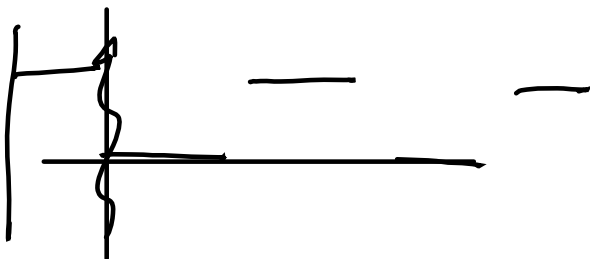
ker je $\left\{ \frac{1}{\sqrt{2\pi}}, \dots \right\}$ ONS velja

$$\lim_{n \rightarrow \infty} a_n = 0 \quad \lim_{n \rightarrow \infty} b_n = 0$$

(Riemann-Lebesgueova lema)

Zagled

$$f(x) = \begin{cases} 1; & 0 \leq x \leq \pi \\ 0; & -\pi < x < 0 \end{cases}$$



$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \\ &= \frac{1}{\pi} \int_0^{\pi} \cos(nx) dx \end{aligned}$$

$$a_0 = 1 \quad n > 0$$

$$a_n = \frac{1}{\pi} \cdot \frac{1}{n} \sin(nx) \Big|_0^{\pi}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \int_0^{\pi} \sin(nx) dx = -\frac{1}{\pi n} \cos(nx) \Big|_0^{\pi} \\ &= \frac{1}{n\pi} (1 - (-1)^n) \end{aligned}$$

$$b_{2k} = 0$$

$$b_{2k+1} = \frac{2}{\pi(2k+1)}$$

KONS:

$$f(x) \stackrel{L^2}{=} \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)} \sin((2k+1)x)$$

≈ 0

Parsevalova enekost

$$\frac{1}{\pi} \pi = 1 = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{4}{\pi^2} \frac{1}{(2k+1)^2}$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$= \frac{\pi^2}{8} + \frac{1}{4}S$$

$$\frac{3}{4}S = \frac{\pi^2}{8}$$

$$S = \frac{\pi^2}{6}$$

Izrek:

Naj bo f odsekoma zvezna in odsekoma odvedljiva periodična funkcija s periodo 2π

1) Na vsakem intervalu dolžine 2π ima največ končna mnogo točk nezveznosti in v vsaki točki obstajata levi in desni limita oznaki

$$x_0 \in \mathbb{R}. \exists \lim_{x \uparrow x_0} f(x) = f(x_0 - 0) = f(x_0^-)$$

$$\exists \lim_{x \downarrow x_0} f(x) = f(x_0 + 0) = f(x_0^+)$$

2) V vsaki točki obstajata levi in desni odvod

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0^+)}{h}$$

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0^-)}{h} = \lim_{h \downarrow 0} \frac{f(x_0 - h) - f(x_0^-)}{(-h)}$$

Torej za $\forall x \in \mathbb{R}$ velja $\frac{f(x^+) + f(x^-)}{2} =$

$$= \frac{a_0}{2} + \sum a_n \cos(nx) + b_n \sin(nx)$$

Pomožte trditelji

1) Něj bo $f: \mathbb{R} \rightarrow \mathbb{R}$ periodická s periodou p odsekom a vzima

Teda je $\forall a \in \mathbb{R}$

$$\int_a^{a+p} f(x) dx = \int_a^p f(x) dx$$

Dokaz:

$$\int_a^{a+p} f(x) dx = \int_a^p f(x) dx + \int_p^{a+p} f(x) dx$$

$$x = t + p$$

$$= \int_a^p f(x) dx + \int_0^a \underset{\parallel}{f(t+p)} dt =$$

$$= \int_a^p f(x) dx + \int_0^a f(t) dt = \int_a^p f(x) dx$$

$$2) \quad \frac{1}{2} + \sum_1^n \cos(kx) = \frac{1}{2} \frac{\sin(n + \frac{1}{2})x}{\sin(\frac{x}{2})} = D_n(x)$$

(Dirichletova jedro)

Dokaz:

$$\frac{1}{2} + \sum_1^n \cos(kx) = \frac{1}{2} + \sum_1^n \frac{e^{ikx} + \bar{e}^{ikx}}{2} =$$

$$= \frac{1}{2} \sum_{j=-n}^n e^{ijx} = \frac{1}{2} e^{-inx} (1 + e^{ix} + e^{2ix} + \dots + e^{2nix})$$

$$= \frac{1}{2} e^{-inx} \frac{1 - e^{(2n+1)ix}}{1 - e^{ix}} =$$

$$= \frac{1}{2} \frac{e^{(n+1)ix} - e^{-inx}}{e^{ix} - 1} = \frac{1}{2} \frac{e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}}{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}}$$

$$= \frac{1}{2} \frac{\sin((n+\frac{1}{2})x)}{\sin \frac{x}{2}}$$

3)

$$\int_{-\pi}^{\pi} D_n(x) dx = \pi$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} D_n(x) dx = 1$$

4) $D_n(x)$ je soda funkcija, 2π periodna, gladka.

$$\begin{aligned}
 5) \quad & \frac{1}{2} \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{x}{2})} = \\
 & = \frac{1}{2} \frac{\sin(nx)\cos\frac{x}{2} + \cos(nx)\sin(\frac{x}{2})}{\sin\frac{x}{2}} = \\
 & = \frac{1}{2} \left(\sin(nx) \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} + \cos(nx) \right)
 \end{aligned}$$

Opombe:

f je lihe na $(-\pi, \pi)$

$$a_n = 0$$

f razvijamo samo po $\{\sin(nx)\}$

f sode na $(-\pi, \pi) \Rightarrow b_n = 0$

f razvijemo le po $\{1, \cos(nx)\}$

f iz $[0, \pi]$ lahko lihe razširimo na $(-\pi, \pi)$

$$x \mapsto \begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}$$

Tako funkcijo ^{lahko} razvijamo le po $\sin(nx)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\int_0^{\pi} f(x) \sin(nx) dx = b_n \int_0^{\pi} \frac{1 - \cos(2nx)}{2} dx = \frac{\pi}{2} b_n$$

novi b_n
↓

Če f razširimo kot sode funkcijo na $(-\pi, \pi)$

$$x \mapsto \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}$$

Tako funkcijo razvijamo po $\{1, \cos(nx)\}$

Opomba 2:

$f \in C^k(\mathbb{R})$ 2π periodična $k \in \mathbb{N}$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x)}_u \underbrace{\cos(nx)}_{dv} dx =$$

$$= \frac{1}{\pi} \left(\underbrace{\frac{1}{n} \sin(nx) f(x)}_{=0} \Big|_{-\pi}^{\pi} - \frac{1}{n} \int_{-\pi}^{\pi} f'(x) \sin(nx) dx \right)$$

= ... naredimo to k krat ... =

$$a_n = -\frac{1}{n\pi} \int_{-\pi}^{\pi} f'(x) \sin(nx) dx$$

$$a_n = -\frac{1}{n\pi} \int_{-\pi}^{\pi} f''(x) \cos(nx) dx$$

⋮

$$a_n = \frac{\pm 1}{n^k} \int_{-\pi}^{\pi} f^{(k)}(x) \underbrace{\cos(nx)}_{\substack{\sin(nx) \\ \uparrow \\ \text{neki od tega}}} dx \quad \text{omejena}$$

$$a_n, b_n = O\left(\frac{1}{n^2}\right)$$

K velikesti: red

večkrat kot je odvedljiva f (kot periodična funkcija) hitreje gredo furierovi koeficienti proti 0

Dde2

$$f \in L^2(-\pi, \pi)$$

$$\sqrt{\int_{-\pi}^{\pi} |f|^2 dx} = \|f\|_2$$

Klasion: funkciji koeficienti:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \quad n = 1, 2, \dots$$

$$f(x) \stackrel{L^2}{=} \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

Parsevalova jednakost

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f|^2 dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2)$$

f periodična s periodom 2π in odsekoma
vezna

• $\forall x_0 \in \mathbb{R}. \exists f(x_0+), f(x_0-)$ limiti in

• odsekoma odvedljiva

• dostajata Levi in Darbouxovi pogoji

$$\text{Potem: } \forall x_0: \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) = \\ = \frac{f(x_0+) + f(x_0-)}{2}$$

Vemo:

• f periodična s periodom p

$$\forall a \in \mathbb{R}. \int_a^{a+p} f(x) dx = \int_a^{a+p} f(x) dx$$

$$\frac{1}{2} + \sum_{k=1}^N \cos(kx) = D_N(x) = \frac{1}{2} \frac{\sin(N + \frac{1}{2}x)}{\sin(\frac{1}{2}x)}$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} D_N(x) dx = 1 \quad \text{so da} \\ \parallel \quad \frac{2}{\pi} \int_0^{\pi} D_N(x) dx$$

Dde2 >>

Dokaz:

$$a_n \cos(nx_0) + b_n \sin(nx_0) =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt \cdot \cos(nx_0) + \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt \cdot \sin(nx_0) =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) (\cos(nt) \cos(nx_0) + \sin(nt) \sin(nx_0)) dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(n(t-x_0)) dt$$

$$\left[\int_{-\pi}^{\pi} f(t) g(x_0-t) dt \dots \text{konvolucija} \right]$$

$$t-x_0=y \quad t=x_0+y$$

$$= \frac{1}{\pi} \int_{-\pi-x_0}^{\pi-x_0} f(x_0+y) \cos(ny) dy =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x_0+y) \cos(ny) dy$$

$$\frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x_0+y) dy = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x_0+y) \cdot \frac{1}{2} dy$$

$$\frac{a_0}{2} + \sum_{k=1}^n a_k \cos(kx_0) + b_k \sin(kx_0) = S_n(x_0)$$

$$S_n(x_0) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x_0+y) \left(\frac{1}{2} \cos y + \dots \cos(ny) \right) dy =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x_0+y) D_n(y) dy =$$

$$= \frac{1}{\pi} \left(\int_0^{\pi} f(x_0+y) D_n(y) dy + \int_{-\pi}^0 f(x_0+y) D_n(y) dy \right)$$

$$= \frac{1}{\pi} \left(\int_0^{\pi} f(x_0+y) D_n(y) dy + \int_0^{\pi} f(x_0-y) D_n(y) dy \right) \quad \leftarrow \text{sada funkcije}$$

$$\xrightarrow{n \rightarrow \infty} \frac{f(x_0+) + f(x_0-)}{2}$$

$$\frac{1}{\pi} \int_0^{\pi} (f(x_0+y) + f(x_0-y)) D_n(y) dy - \frac{f(x_0+) + f(x_0-)}{2}$$

$$\frac{2}{\pi} \cdot \int_0^{\pi} D_n(y) dy \quad \parallel \quad 1$$

$$= \frac{1}{\pi} \int_0^{\pi} (f(x_0+y) - f(x_0+)) D_n(y) + (f(x_0-y) - f(x_0-)) D_n(y) dy$$

$$\frac{1}{2} \int_0^{\pi} (f(x_0+y) - f(x_0+)) \frac{\sin(n+\frac{1}{2})y}{\sin(\frac{y}{2})} dy =$$

$$= \frac{1}{2} \int_0^{\pi} (f(x_0+y) - f(x_0+)) \frac{\cos(\frac{y}{2})}{\sin(\frac{y}{2})} \sin(ny) + \cos(ny) dy$$

Pogledamo sumand

$$\int_0^{\pi} (f(x_0+y) - f(x_0+)) \cos(ny) dy \xrightarrow{n \rightarrow \infty} 0$$

$$F(y) = \begin{cases} f(x_0+y) - f(x_0+) & ; 0 \leq y \leq \pi \\ 0 & ; -\pi < y < 0 \end{cases}$$

odaklema

Wierze

Riemann-Lebesgue lemma

$$= \int_{-\pi}^{\pi} F(y) \cos(ny) dy \xrightarrow{n \rightarrow \infty} 0$$

Ge

$$\int_0^{\pi} f(x_0+y) - f(x_0+) \frac{\cos(\frac{y}{2})}{\sin(\frac{y}{2})} \sin(ny) dy$$

$$= \int_0^{\pi} \frac{f(x_0+y) - f(x_0+)}{y} \frac{y}{\sin(\frac{y}{2})} \cos(\frac{y}{2}) \sin(ny) dy$$

$$G(y) = \begin{cases} \frac{f(x_0+y) - f(x_0+)}{y} \frac{y}{\sin(\frac{y}{2})} \cdot \cos(\frac{y}{2}) & ; 0 < y \leq \pi \\ 0 & ; -\pi < y < 0 \end{cases}$$

$$\int_{-\pi}^{\pi} G(y) \sin(ny) dy \quad \text{za } y \neq 0 \text{ je } f \text{ odsekoma}$$

zvezna

za $y=0$ pa obstaja limit

$$\lim_{y \rightarrow 0} G(y) = G$$

$$\lim_{y \rightarrow 0} G(y) = 2 \lim_{y \rightarrow 0} \frac{f(x_0+y) - f(x_0+)}{y}$$

Riemann-Lebesgue lemma:

obstaja

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} G(y) \sin(ny) dy = 0$$

po predpostavki.

za drugi sumand uporabimo podobne argumente

Zagled:

$$p(x) = \begin{cases} 1 & 0 \leq x \leq \pi \\ 0 & -\pi < x < 0 \end{cases}$$

Furierova vrsta: $\frac{1}{2} + \frac{2}{\pi} \sum_0^{\infty} \frac{1}{2k+1} \sin(2k+1)x$



$$\frac{f(0+) + f(0-)}{2} = \frac{1}{2}$$

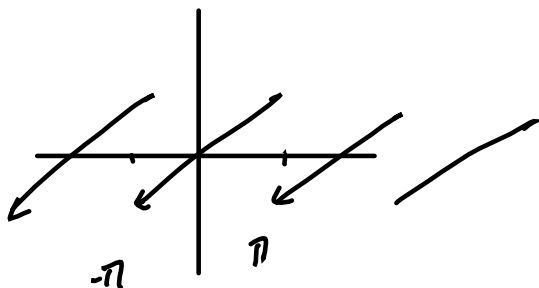
$$x = \frac{\pi}{2}: \quad 1 = \frac{1}{2} + \frac{2}{\pi} \sum_0^{\infty} \frac{1}{2k+1} \sin(2k+1)\frac{\pi}{2}$$

↑
+1 al: $p_a - 1$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

zaged

$$f(x) = x$$



$$a_n = 0 \quad \forall n$$

ker je na $(-\pi, \pi)$ funkcija l.h.e

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx$$

$$u = x \quad dv = \sin(nx) \quad n \geq 1$$

$$du = dx \quad v = -\frac{1}{n} \cos(nx)$$

$$b_n = \frac{1}{\pi} \left(-\frac{x}{n} \cos(nx) \right) \Big|_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} \cos(nx) dx$$

$\underbrace{\hspace{10em}}_{=0}$

$$= \frac{1}{\pi} \left(-\frac{\pi}{n} \cos(n\pi) + \frac{(-\pi)}{n} \cos(n\pi) \right) =$$

$$= -\frac{2}{n} (-1)^n = \frac{2}{n} (-1)^{n+1}$$

$$x \stackrel{L^2(-\pi, \pi)}{=} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin(nx)$$

Po točki:

$$x=0 : 0=0$$

$$x=\pi : \frac{\pi + (-\pi)}{2} = 0 \quad \checkmark$$

$$x=\frac{\pi}{2} : \frac{\pi}{2} = 2 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right)$$

Parsevalova enakost

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \sum_{n=1}^{\infty} \frac{4}{n^2} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

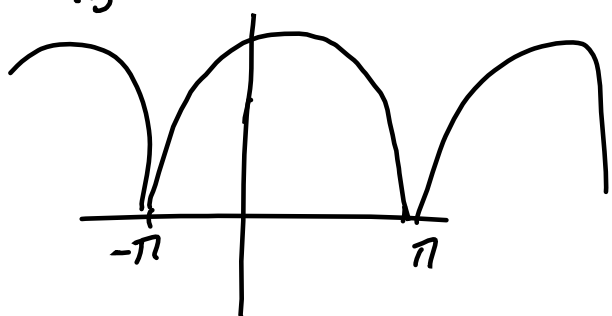
$\parallel \quad \uparrow$
rekvadrat

$$\frac{1}{\pi} \left(\frac{1}{3} x^3 \right) \Big|_{-\pi}^{\pi} = \frac{2}{\pi} \pi^3 \cdot \frac{1}{3} = \frac{2}{3} \pi^2 = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Zaplel.

$$f(x) = \pi^2 - x^2$$



soda, nenegativno
zveze, v vsaki
točki ima levi in
desni odvod

$$\text{sodast} \Rightarrow b_n = 0$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi^2 - x^2) dx = \frac{2}{\pi} \left(\pi^3 - \frac{1}{3} \pi^3 \right) = \frac{4}{3} \pi^2$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{(\pi^2 - x^2)}_u \underbrace{\cos(nx)}_{dv} dx =$$

$$= \frac{1}{\pi} \left(\pi^2 - x^2 \right) \frac{1}{n} \sin(nx) \Big|_{-\pi}^{\pi} + \frac{1}{\pi} \int_{-\pi}^{\pi} 2x \sin(nx) dx =$$

$$= \frac{2}{\pi n} \int_{-\pi}^{\pi} x \sin(nx) dx =$$

$$= \frac{2}{\pi n} \left(x \left(-\frac{1}{n} \cos(nx) \right) \right) \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \cos(nx) dx =$$

$$= \frac{2}{\pi n^2} \left(\pi (-1)^{n+1} + (-1)^{n+1} \pi \right) = \frac{4}{n^2} (-1)^{n+1}$$

$$f(x) = \frac{2}{3\pi^2} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^{n+1} \cos(nx) \quad \forall x \in \mathbb{R}$$

$$x=0 \Rightarrow$$

$$f(0) = \pi^2 = \frac{2}{3} \pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^{n+1}$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

$$x=\pi \Rightarrow$$

$$0 = \frac{2}{3} \pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^{n+1} (-1)^n$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

$$\text{kons: } \pi$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} (\pi^2 - x^2)^2 dx = \frac{1}{\pi} \left(\pi^4 x - \frac{2}{3} \pi^2 x^3 + \frac{1}{5} x^5 \right) \Big|_{-\pi}^{\pi} = \frac{8\pi^4}{15}$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} (\pi^4 - 2x^2 \pi^2 + x^4) dx = \frac{8\pi^4}{15} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{2}{\pi} \left(\pi^5 - \frac{2\pi^5}{3} + \frac{1}{5} x^5 \right) = \frac{8\pi^4}{15} + 16S$$

$$\frac{15 - 10 + 3}{15} \pi^4 = \frac{4\pi^4}{15} + 8S$$

$$\frac{8\pi^4}{15}$$

$$8S = \frac{8\pi^4}{15} - \frac{4\pi^4}{15}$$

$$2S = \frac{2\pi^4}{15} - \frac{\pi^4}{15} = \frac{6\pi^4}{45} - \frac{5\pi^4}{45} = \frac{\pi^4}{45}$$

$$S = \frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$

$$a_1, a_2, \dots \quad \lim_{n \rightarrow \infty} a_n = a$$

$$1, -1, 1, -1$$

$$\frac{1}{1}, \frac{1-1}{2}, \frac{1-1+1}{3} \Rightarrow 0$$

$$a_1, \frac{a_1+a_2}{2}, \frac{a_1+a_2+a_3}{3}$$

konverģence $\Rightarrow$ konverģence + paritēti

$$S_n(x_0) = \int_{-\pi}^{\pi} f(x_0 + y) D_n(y) dy$$

$$\frac{S_0(x) + S_1(x) + \dots + S_{n-1}(x)}{n} = \sigma_n(x) \xrightarrow{n \rightarrow \infty} f$$

če sárove delne vsake

če je f zveza je kar se kaže
enakomerno

Fejérjevo jedro: $F_N(x) = \frac{1}{N} \sum_{n=0}^{N-1} D_n(x)$

Trditve:

1) $F_N(x) = \frac{1}{2N} \left(\frac{\sin(\frac{Nx}{2})}{\sin(\frac{x}{2})} \right)^2$

2) F_N je sode

3) $F_N(x) \geq 0$ za $\forall x$

4) $\frac{1}{\pi} \int_{-\pi}^{\pi} F_N(x) dx = 1$

5) $\forall a, 0 < a < \pi, \lim_{N \rightarrow \infty} F_N(x) = 0$
enakomerno na $a \leq |x| \leq \pi$

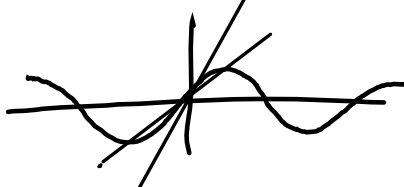
Dokaz:

1) $\Rightarrow$ 2) očitno

1) $\Rightarrow$ 3) očitno

iz definicije $\Rightarrow$ 4) $\frac{1}{\pi} \int_{-\pi}^{\pi} D_n(x) dx = 1$

1) $\Rightarrow$ 5)



$\forall y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ velja $\frac{2}{\pi} |x| \leq |\sin y|$

$0 < a \leq |x| \leq \pi$

$0 < \frac{a}{2} \leq |\frac{x}{2}| \leq \frac{\pi}{2}$

$\frac{2}{\pi} \cdot \frac{|x|}{2} \leq |\sin(\frac{x}{2})|$

$\frac{1}{|\sin \frac{x}{2}|} \leq \frac{\pi}{|x|} \leq \frac{\pi}{a}$

$F_N(x) = \frac{1}{2N} \frac{|\sin(\frac{Nx}{2})|^2}{|\sin(\frac{x}{2})|^2} \leq \frac{1}{2N} \cdot \frac{\pi^2}{a^2}$

$\lim_{N \rightarrow \infty} F_N(x) = 0$ enakomerno na $a \leq |x| \leq \pi$

1) $F_N(x) = \frac{1}{N} \sum_{n=0}^{N-1} D_n(x) = \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{2} \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{x}{2})} =$

$= \frac{1}{2N} \frac{1}{\sin^2(\frac{x}{2})} \sum_{n=0}^{N-1} \sin(n+\frac{1}{2})x \cdot \sin(\frac{x}{2}) =$

$= \frac{1}{2N \sin^2(\frac{x}{2})} \frac{1}{2} \sum_{n=0}^{N-1} (\cos(nx) - \cos((n+1)x))$

$(1 - \cos x) + (\cos x - \cos 2x) + (\cos 2x - \cos 3x) + \dots$

$= 1 - \cos(Nx)$

$= \frac{1 - \cos(Nx)}{4N \sin^2(\frac{x}{2})} = \frac{1}{2N} \frac{\sin^2(\frac{Nx}{2})}{\sin^2(\frac{x}{2})}$

breki:

Naj bo f zveza 2π periodična funkcija
Potem cez ravne delne vsote

$$\sigma_N(x) = \frac{1}{N} (S_0(x) + \dots + S_{N-1}(x))$$

konvergirajo k f enakomerno na

$[-\pi, \pi]$ oziroma na $\mathbb{R}$

$$S_N(x) = a_0 + \sum_{k=1}^N (a_k \cos(kx) + b_k \sin(kx))$$

T : geometrični polinom... lineare
kombinacije sinusov in kosinusov

Dokaz:

$$\sigma_N(x) = \frac{1}{N} (S_0(x) + \dots + S_{N-1}(x))$$

$$S_N(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+y) D_N(y) dy$$

$$\sigma_N(x) = \frac{1}{N\pi} \int_{-\pi}^{\pi} f(x+y) F_N(y) dy \xrightarrow[\text{---}]{\text{enakomerno}} f(x)$$

Naj bo $\varepsilon > 0$. Ker je f zveza in periodična
je enakomerno zveza

$$\exists \delta > 0. \forall y \in \mathbb{R}. |f(x+y) - f(x)| < \frac{\varepsilon}{2}$$

$$|\sigma_N(x) - f(x)| = \left| \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+y) F_N(y) dy - f(x) \frac{1}{\pi} \int_{-\pi}^{\pi} F_N(y) dy \right|$$

$$= \left| \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x+y) - f(x)) F_N(y) dy \right| \leq$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x+y) - f(x)| F_N(y) dy =$$

$$= \frac{1}{\pi} \int_{-\delta}^{\delta} |f(x+y) - f(x)| F_N(y) dy + \int_{\delta < |y| \leq \pi} |f(x+y) - f(x)| F_N(y) dy$$

$$< \frac{1}{\pi} \int_{-\delta}^{\delta} F_N(y) dy \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2}$$

$\rightarrow f$ je zveza in periodična torej je omejena

$$\exists M \in \mathbb{R} \quad |f(z)| \leq M \quad \forall z \in \mathbb{R}$$

$$\Rightarrow f(x+y) - f(x) \leq 2M \quad \forall x, y \in \mathbb{R}$$

$$\frac{1}{\pi} \int_{\delta < |y| \leq \pi} |f(x+y) - f(x)| F_N(y) dy \leq 2M \frac{1}{\pi} \int_{\delta < |y| \leq \pi} F_N(y) dy$$

$$\text{Vemo da je } \lim_{N \rightarrow \infty} F_N(y) = 0$$

enakomerno na $\delta < |y| \leq \pi$

$$\exists N_0. \forall N \geq N_0. |F_N(y)| \leq \frac{\varepsilon}{2M \cdot 4} = \frac{\varepsilon}{8M}$$

$$\frac{2M}{\pi} \int_{\delta < |y| \leq \pi} F_N(y) dy \leq \frac{2M}{\pi} \cdot \frac{\varepsilon}{8M} \cdot 2\pi = \frac{\varepsilon}{2}$$

$\forall N \geq N_0$ je

$$|\sigma_N(x) - f(x)| < \varepsilon \quad \forall x \in \mathbb{R}$$

traj:

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx); n \in \mathbb{N} \right\}$$

je KONS v $L^2(-\pi, \pi)$

Dokaz:

$L^2(-\pi, \pi)$ je nadpolnster $C([-\pi, \pi])$ v

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) dx$$

(ONS je KONS $\Leftrightarrow$ končne linearne kombinacije vektorjev iz ONS goste v prostoru

OPOMBA: KONČNE LINEARNE KOMBINACIJE

$\sin(nx), \cos(nx), n \in \mathbb{N}_0$ so trigonometrični polinom:

Ali so trigonometrični polinomi gosti v $L^2(-\pi, \pi)$?

1) $C(-\pi, \pi)$ so goste v $L^2(-\pi, \pi)$

(Priznamo)

$$f \in L^2(-\pi, \pi). \varepsilon > 0. \exists \tilde{f} \in C[-\pi, \pi]. \|f - \tilde{f}\| < \frac{\varepsilon}{2}$$

2) Ali so trigonometrični polinomi gosti v $(C[-\pi, \pi], d_2)$?

če da:

$\exists T(x)$ trigonometrični polinom

$$T(x) = x_0 + \sum_{j=1}^N \lambda_j \cos(jx) + \mu_j \sin(jx)$$

$$\|\tilde{f} - T\|_2 < \frac{\varepsilon}{2} \Rightarrow \|f - T\|_2 < \varepsilon$$

OPOMBA:

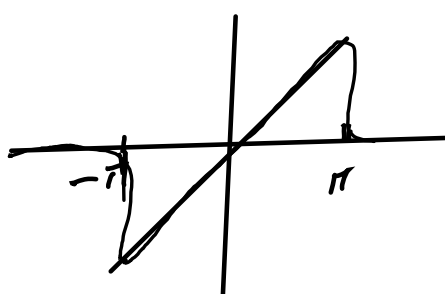
$$g_n \in L^2[-\pi, \pi] \xrightarrow[n \in \mathbb{N}]{\text{enakomerno}} f$$

$$\Rightarrow \lim_{n \rightarrow \infty} g_n = f \text{ v } d_2$$

$$d_2(f, g_n)^2 = \int_{-\pi}^{\pi} (f - g_n)^2 \xrightarrow{n \rightarrow \infty} 0$$

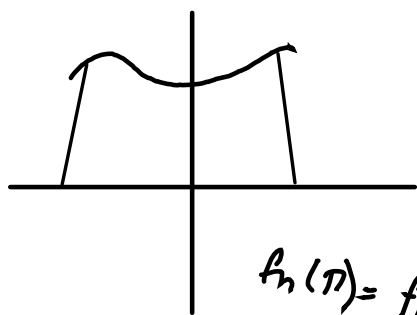
Vemo: f zveza 2π periodična $\Rightarrow$ jo lahko poljubno dobro enakomerno aproksimiramo s trigonometričnimi polinomi:

$$\|f - \sigma_n\|_{\infty} < \varepsilon$$



n : zveza 2π periodična funkcije

$$f_n(x) = \begin{cases} f(x) & \\ -n(f(\pi - \frac{1}{n})(x - \pi)); & \pi - \frac{1}{n} < x \leq \pi \\ n f(-\pi + \frac{1}{n})(x + \pi) & -\pi \leq x < -\pi + \frac{1}{n} \end{cases}$$



$$f_n(\pi) = f_n(-\pi) = 0$$

zveza 2π periodična funkcije

$$\|f - f_n\|_2^2 = \int_{-\pi}^{\pi} (f_n(x) - f(x))^2 dx \leq \frac{2}{n} (2\pi)^2 \xrightarrow{n \rightarrow \infty} 0$$

Isrek: (Weistrassov)

Naj bo f wena na $[a, b]$

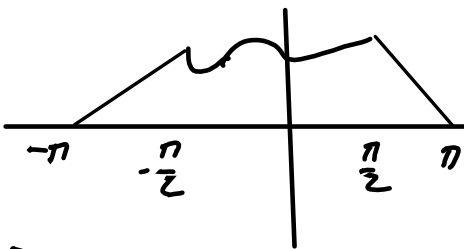
Naj bo $\epsilon > 0$. Potem $\exists$ polinom p da je

$$\|f - p\|_{\infty} < \epsilon$$

Dokaz: Dovolj je opazeti $a = -\frac{\pi}{2}$ $b = \frac{\pi}{2}$

f wena na $[-\frac{\pi}{2}, \frac{\pi}{2}]$

f wena razširimo na $[-\pi, \pi]$



$$\tilde{f} = \begin{cases} f(x) & x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ -\frac{2}{\pi} f(\frac{\pi}{2})(x - \pi) & \frac{\pi}{2} < x \leq \pi \\ \frac{2}{\pi} f(-\frac{\pi}{2})(x + \pi) & -\pi \leq x < -\frac{\pi}{2} \end{cases}$$

$\tilde{f}$ je wena 2π periodna.

za $\epsilon > 0 \exists T$ trigonometrični polinom, da je $\|\tilde{f} - T\|_{\infty} < \frac{\epsilon}{2}$

$\cos(4x)$ $\sin(4x)$ jih je končno

Approxim: samo s Taylorjevimi polinom: enakomerno.

Vektorska analiza (v $\mathbb{R}^2$ ali $\mathbb{R}^3$)

$D \subseteq \mathbb{R}^3$ $u: D \rightarrow \mathbb{R}$ ~~vektor~~ funkcija
jo imenujemo skalarno polje

vsaki točki v D priredi skalar

$D \subseteq \mathbb{R}^3$ $\vec{R}: D \rightarrow \mathbb{R}^3$ ~~vektor~~

vektorsko polje vsaki točki priredi
vektor $\vec{R}(t)$

$\mathbb{R}^3$

$u(x, y, z) = x$ v bazi $\vec{e}_1, \vec{e}_2, \vec{e}_3$ ($\vec{i}, \vec{j}, \vec{k}$)

Baza: $\vec{p} = \frac{1}{\sqrt{2}} \vec{e}_1 + \frac{1}{\sqrt{2}} \vec{e}_3$

$\vec{q} = -\frac{1}{\sqrt{3}} \vec{e}_1 + \frac{1}{\sqrt{3}} \vec{e}_2 + \frac{1}{\sqrt{3}} \vec{e}_3$

$\vec{r} = \frac{1}{\sqrt{6}} \vec{e}_1 + \frac{2}{\sqrt{6}} \vec{e}_2 - \frac{1}{\sqrt{6}} \vec{e}_3$

Točka (x, y, z) ima v Bazi $(\vec{p}, \vec{q}, \vec{r})$

koordinata (α, β, γ)

KOORD

$(\alpha, \beta, \gamma) \mapsto \alpha$ ni ista funkcija

ONB je pozitivno orientirana, če je
 $[\vec{p}, \vec{q}, \vec{r}] > 0 \Leftrightarrow \vec{r} = \vec{p} \times \vec{q}$

oziroma negativno orientirana, če

$$[\vec{p}, \vec{q}, \vec{r}] < 0 \Leftrightarrow \vec{r} = -\vec{p} \times \vec{q}$$

(oznake od zdej naprej za mešani
 produkt bo $(\vec{p}, \vec{q}, \vec{r})$ najbrž

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} = U \quad \begin{matrix} (x, y, z) \quad (e_1, \dots, e_n) \\ (\alpha, \beta, \gamma) \quad (\vec{p}, \vec{q}, \vec{r}) \end{matrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = U \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$U^T U = I$$

$$U^{-1} = U^T$$

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = U^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$u(x, y, z)$ v (e_1, e_2, e_3) koordinatah

$$\tilde{u}(\alpha, \beta, \gamma) = u(U \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix})$$

Vektoren $\vec{p}$

$$\vec{R}(x,y,z) = (X(x,y,z), Y(x,y,z), Z(x,y,z))$$

$U(\alpha, \beta, \gamma)$ Koordinaten

$$\tilde{R}(\alpha, \beta, \gamma) = (U^T \circ \vec{R} \circ U) \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$\tilde{u} = \frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{3}} + \frac{\gamma}{\sqrt{6}}$$

$$\vec{R}(x,y,z) = (x+2y+3z, x+2y+3z, x+2y+3z)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$x = \frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{3}} + \frac{\gamma}{\sqrt{6}}$$

$$y = \frac{\beta}{\sqrt{3}} + \frac{2\gamma}{\sqrt{6}}$$

$$z = \frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{3}} - \frac{\gamma}{\sqrt{6}}$$

$$x+2y+3z = \frac{4}{\sqrt{2}}\alpha + \frac{4}{\sqrt{3}}\beta + \frac{2}{\sqrt{6}}\gamma = L$$

$$\text{to ist } (R \circ U) \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = (L, L, L)$$

$$U(R \circ U \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}) = \cancel{L} \left(\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{6}} \right) L$$

V

neh: pred
3.3

maže

$D \subset \mathbb{R}^3$ povezljiv

$$u: D \rightarrow \mathbb{R} \quad C^1$$

$$u \mapsto \vec{\nabla} u = \text{grad} u = (u_x, u_y, u_z)$$

$$\vec{R}: D \rightarrow \mathbb{R}^3 \quad \vec{R} = (X, Y, Z)$$

$$\vec{R} \mapsto \text{div} \vec{R} = \vec{\nabla} \cdot \vec{R} = X_x + Y_y + Z_z$$

$$\vec{R} \mapsto \text{rot} \vec{R} = \vec{\nabla} \times \vec{R} = (Z_y - Y_z, X_z - Z_x, Y_x - X_y)$$

$$u \in C^2(D), \vec{R}: D \rightarrow \mathbb{R}^3 \quad C^2$$

$$\text{rot grad} u = \vec{0}$$

$$\vec{\nabla} \times \vec{\nabla} u = \vec{0}$$

$$\text{div rot} \vec{R} = 0$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{R}) = 0$$

$$\text{div grad} u = \vec{\nabla} \cdot \vec{\nabla} u = u_{xx} + u_{yy} + u_{zz} = \Delta u$$

Izrek:

D nezdasta (7 točka, ki vidi vse ostale točke )

$$\vec{R}: D \rightarrow \mathbb{R}^3 \quad \text{vek polje}$$

$$1) \text{ rot } \vec{R} = \vec{0} \Rightarrow \exists u \in C^2(D). \vec{R} = \vec{\nabla} u$$

$$2) \text{ div } \vec{R} = 0 \Rightarrow \exists \vec{G}: D \rightarrow \mathbb{R}^3 \quad C^2 \text{ vek polje}$$

da je $\vec{R} = \text{rot } \vec{G}$

Zajed

da 1) ne velja obratno

$$\mathbb{R}^3 - z\text{-os} \quad D: \text{ nezdasta}$$

$$\vec{R} = \left(\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$$

$$\arctan \frac{y}{x} = u \quad \text{ni dobro definirana na}$$

1) če ima $\vec{R}$ potencial u , poten so vsi potenciali oblike $u + C$ (D povezan)

Reč

$$\vec{R} : D \rightarrow \mathbb{R}^3 \text{ vec. polje}$$

$$\operatorname{div} \vec{R} = f \quad f \in C^1(D)$$

$$\exists u \text{ na } D, \Delta u = f$$

$$\operatorname{div} \operatorname{grad} u = \Delta u = f = \operatorname{div} \vec{R}$$

$$\operatorname{div} (\vec{R} - \operatorname{grad} u) = 0$$

$$\parallel$$

$$\operatorname{rot}(\vec{G})$$

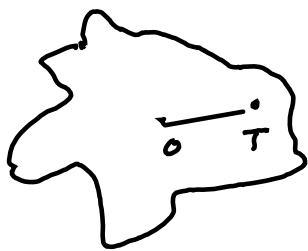
$$\vec{R} = \operatorname{grad} u + \operatorname{rot} \vec{G}$$

Dokaži (izveďa dve strani: nezej)

$$1) \vec{R}: D \rightarrow \mathbb{R}^3 \quad C^1(D)$$

$$\operatorname{rot} \vec{R} = \vec{0} = (z_y - y_z, x_z - z_x, y_x - x_y) = \vec{0}$$

BŠZS. je D zvezdasto glede na o



če $(x, y, z) \in D$.

$$\forall t \in [0, 1], (tx, ty, tz) \in D$$

Definiramo:

$$u(x, y, z) = \int_0^1 (X(tx, ty, tz) \cdot x + Y(tx, ty, tz) \cdot y + Z(tx, ty, tz) \cdot z) dt$$

u je potencial za $\vec{R}$

$$\underline{u_x = X \quad u_y = Y \quad u_z = Z}$$

$$\begin{aligned} u_x &= \int_0^1 (X(tx, ty, tz) + \underbrace{X_x \cdot t \cdot x}_{\parallel X_y} + \underbrace{Y_x \cdot ty}_{\parallel X_z} + Z_x \cdot tz) dt \\ &= \int_0^1 (X(tx, ty, tz) + tx \cdot X_x + ty \cdot X_y + tz \cdot X_z) dt \\ &= \frac{d}{dt} (t \cdot X(tx, ty, tz)) \end{aligned}$$

$$= t X(tx, ty, tz) \Big|_0^1 = X(x, y, z)$$

Podobno $u_y = Y$ iz $u_z = Z$

Dleaz

2)

$$\operatorname{div} \vec{R} = \vec{0} = X_x + Y_y + Z_z$$

(D je nečista gledi na 0 BŠZS)

$$\alpha(x, y, z) = \int_0^1 t X(tx, ty, tz) dt$$

$$\beta(x, y, z) = \int_0^1 t Y(tx, ty, tz) dt$$

$$\gamma(x, y, z) = \int_0^1 t Z(tx, ty, tz) dt$$

$$\operatorname{div}(\alpha, \beta, \gamma) = 0 \text{ ker:}$$

$$\alpha_x = \int_0^1 t^2 X_x(tx, ty, tz) dt$$

$$\beta_y = \int_0^1 t^2 Y_y(tx, ty, tz) dt$$

$$\gamma_z = \int_0^1 t^2 Z_z(tx, ty, tz) dt$$

} +

$$= 0$$

$$\vec{G} = \begin{pmatrix} \alpha & \beta & \gamma \\ x & y & z \end{pmatrix} = (\alpha, \beta, \gamma) \times (x, y, z)$$

$$\vec{G} = (z\beta - y\gamma, x\gamma - z\alpha, y\alpha - x\beta)$$

$$\vec{R} = \operatorname{rot} \vec{G} ?$$

.....
1 komponenta od $\operatorname{rot} \vec{G}$:

$$(y\alpha - x\beta)_y - (x\gamma - z\alpha)_z =$$

$$= \alpha + y\alpha_y - x\beta_y - x\gamma_z + \alpha + z\alpha_z$$

$$= 2\alpha + \alpha_x + y\alpha_y + z\alpha_z$$

↳ upotrebimo pravocrtan $\alpha_x + \beta_y + \gamma_z = 0$

$$= \int_0^1 \underbrace{(2tX(tx, ty, tz) + x t^2 X_x + y t^2 X_y + z t^2 X_z)}_{\frac{d}{dt} (t^2 X(tx, ty, tz))} dt$$

$$t^2 X(tx, ty, tz) \Big|_0^1 = X(x, y, z)$$

Podobno se drug. dve komponente:

Zgled:

$$\vec{R} = (y^2 z^3 + 2, 2xy z^3 + 1, 3xy^2 z^2)$$

$$\text{rot } \vec{R} \begin{pmatrix} 6xy z^2 - 6xy z^2 \\ 3y^2 z^2 - 3y^2 z^2 \\ 2yz^3 - 2yz^3 \end{pmatrix} = \vec{0}$$

$$u(x, y, z) = \int_0^1 X(tx, ty, tz) (x + \dots) dt$$

u potencial obzajr: $u_x = y^2 z^3 + 2$

$$u_y = 2xy z^3 + 1$$

$$u_z = 3xy^2 z^2$$

$$u(x, y, z) = xy^2 z^3 + 2x + C(y, z)$$

$$\downarrow$$

$$u_y = 2xy z^3 + C_y = 2xy z^3 + 1$$

$$C_y = 1$$

$$C(y, z) = y + D(z)$$

$$u(x, y, z) = xy^2 z^3 + 2x + y + D(z)$$

$\downarrow$

$$u_z = 3xy^2 z^2 + D'(z) = 3xy^2 z^2$$

$$D' = 0$$

$$D = D_0 \text{ konstanta}$$

$$u(x, y, z) = xy^2 z^3 + 2x + y + D_0$$

($\text{Rot } \vec{R} \neq \vec{0}$ torej ni potenciala)

$$u = xy^2 z^3 + 2x + A(yz)$$

$\downarrow$

$$2xy z^3 + Ay = 2yz^3 + x$$

$Ay = x$ * ker je A funkcija
y in zja nima

Zufall

$$\vec{R} = (2y - 1, -1, 4x - 2xy)$$

$$\operatorname{div} \vec{R} = (0, 0, 0) = 0$$

is there $\vec{G} = ?$ $\operatorname{rot} \vec{G} = \vec{R}$

$$a(x, y, z) = \int_0^1 (2t + y - 1) dt = \frac{2}{3}y - \frac{1}{2}$$

$$b(x, y, z) = \int_0^1 (-1) dt = -\frac{1}{2}$$

$$c(x, y, z) = \int_0^1 (4t + x - 2xyt^2) dt = \frac{4x}{3} - \frac{1}{2}xy$$

$$\vec{G} = \begin{pmatrix} \frac{2}{3}y - \frac{1}{2} & -\frac{1}{2} & \frac{4}{3}x - \frac{1}{2}xy \\ x & y & z \end{pmatrix} =$$

$$= \left(-\frac{1}{2}z - \frac{4}{3}xy + \frac{1}{2}xy^2, \frac{4}{3}x^2 - \frac{1}{2}xy^2 - \frac{2}{3}zy + \frac{1}{2}z, \frac{2}{3}y^2 - \frac{1}{2}y + \frac{1}{2}x \right)$$

$$\operatorname{rot} \vec{G} = \begin{pmatrix} \frac{4}{3}y - \frac{1}{2} + \frac{2}{3}y - \frac{1}{2} & -\frac{1}{2} - \frac{1}{2} & \frac{8}{3}x - xy + \frac{4}{3}x - xy \\ 2y - 1 & -1 & = 4x - 2xy \end{pmatrix}$$

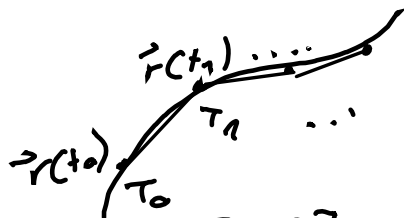
Krivuljni integral

Dužina krivulje:

Γ neka C^1 krivulja u $\mathbb{R}^3$

reg. Parametrizacija $\vec{r}(t) : \vec{r}[\alpha, \beta] \rightarrow \Gamma$

$$\dot{\vec{r}}(t) \neq 0$$



$\mathcal{D}$ delitev $[\alpha, \beta]$

$$\sum_1^n d(T_{j-1}, T_j) = l(\mathcal{D})$$

$$l(\mathcal{D}) = \sum_1^n \sqrt{(x(t_j) - x(t_{j-1}))^2 + \dots + (z(t_j) - z(t_{j-1}))^2}$$

$$\stackrel{\text{poznajmo}}{=} \sum_1^n \sqrt{\dot{x}^2(\xi_j) + \dot{y}^2(\xi_j) + \dot{z}^2(\xi_j)}$$

$$\doteq \sum_1^n \sqrt{\dot{x}^2(\xi_j) + \dot{y}^2(\xi_j) + \dot{z}^2(\xi_j)} \doteq$$

$$= R(\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}, \mathcal{D}, \tau) \xrightarrow{\max \Delta t_j \rightarrow 0}$$

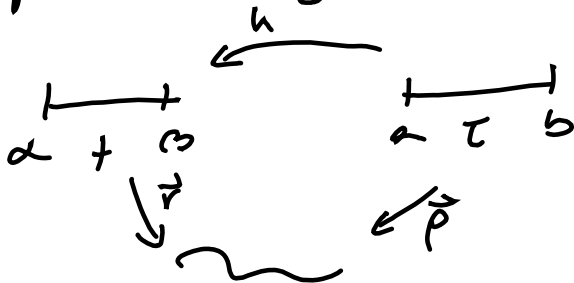
$$\rightarrow \int_{\alpha}^{\beta} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt$$

Naj bo Γ C^1 krivulja v $\mathbb{R}^3$ z regularno parametrizacijo $\vec{r}: [\alpha, \beta] \rightarrow \Gamma$

Definiramo dolžino krivulje

$$l(\Gamma) = \int_{\alpha}^{\beta} |\vec{r}'| dt = \int_{\alpha}^{\beta} \sqrt{x^2 + y^2 + z^2} dt$$

Ali je ta definicija neodvisna od regularne parametrizacije



$$\vec{h}: [a, b] \xrightarrow{D/F} [\alpha, \beta]$$

$$\vec{p} = \vec{r} \circ \vec{h}$$

$$\vec{p}' = \vec{r}'(h) h'$$

$$\int_a^b |\vec{p}'| d\tau = \int_{\alpha}^{\beta} |\vec{r}'| dt$$

$$\int_{\alpha}^{\beta} |\vec{r}'(t)| dt = \int_{[\alpha, \beta]} |\vec{r}'(t)| dt = \int_{[a, b]} |\vec{r}'(h(\tau))| |h'(\tau)| d\tau$$

$t = h(\tau)$

$$= \int_{[a, b]} |\vec{p}'| d\tau$$

Γ naj bo C^1 krivulja $\vec{r}(t)$ reg. prem.

$$S(t) = \int_{\alpha}^t |\dot{\vec{r}}(\tau)| d\tau \quad S: [\alpha, \beta] \rightarrow [0, l(\Gamma)]$$

= dolžina Γ med $\vec{r}(\alpha)$ in $\vec{r}(t)$

odvod: $\dot{S} = |\dot{\vec{r}}(t)| > 0$

S je strogo naraščajoča od $[\alpha, \beta] \rightarrow [0, l(\Gamma)]$

torej ima inverz:

$$T: [0, l(\Gamma)] \rightarrow [\alpha, \beta]$$

$$s \mapsto T(s) \quad \text{inverz od } S$$

Poglejmo parametrizacijo Γ :

$$s \mapsto \vec{r}(T(s))$$

$$\text{Vemo } S \circ T = \text{id}_{[0, l(\Gamma)]} \text{ in } T \circ S = \text{id}_{[\alpha, \beta]}$$

odvajamo:

$$\dot{S}(T(s)) \cdot T'(s) = 1$$

izračunajmo velikost parametrizacije

$$s \mapsto \vec{r}(T(s))$$

$$\frac{d}{ds} |\vec{r}(T(s))| = \dot{\vec{r}}(T(s)) \cdot T'(s) =$$

$$= \frac{\dot{\vec{r}}(T(s))}{\dot{S}(T(s))} = \frac{\dot{\vec{r}}(T(s))}{|\dot{\vec{r}}(T(s))|}$$

velikost hitosti je 1

Tako parametrizirana krivulja je
naravno parametrizirana in S je
naravni parameter

Zgled:

vij = eliptična:

$$t \mapsto (a \cos t, a \sin t, bt) = \vec{r}(t) \quad \begin{matrix} a > 0 \\ b \neq 0 \end{matrix}$$

$$|\dot{\vec{r}}| = \sqrt{a^2 + b^2}$$

$$S(t) = \int_0^t \sqrt{a^2 + b^2} \, dI = \sqrt{a^2 + b^2} \cdot t = s$$

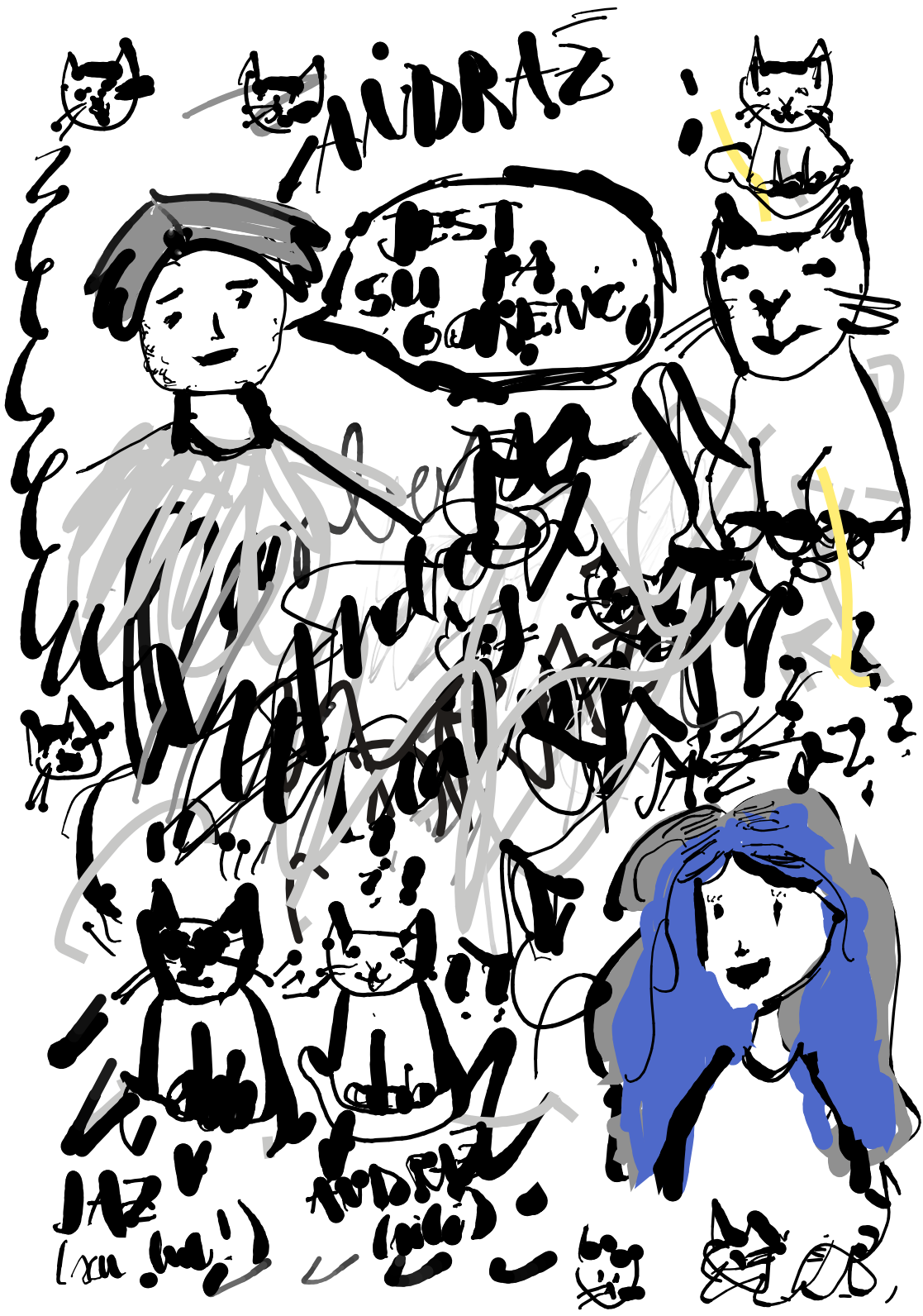
$$T(s) = \frac{s}{\sqrt{a^2 + b^2}}$$

neravno parametrizirane vijčnice

$$s \mapsto \left(a \cos \left(\frac{s}{\sqrt{a^2 + b^2}} \right), a \sin \left(\frac{s}{\sqrt{a^2 + b^2}} \right), \frac{bs}{\sqrt{a^2 + b^2}} \right)$$

$$\| \vec{T}(s) \|$$

$$|\vec{T}_N(s)| = 1$$



Orientacija krivulje



Γ gladka C^1 krivulja

Orientacija Γ je zvezen izbor
enotskega tangentnega vektorja
vzdolž Γ : $\vec{T}$

$$|\vec{T}| = 1$$

Γ povezane $\Rightarrow \Gamma$ ima dve orientaciji:
 $\vec{T}$ in $-\vec{T}$

$\vec{r}: [\alpha, \beta] \rightarrow \Gamma$ reg. parametrizacija

$$\vec{T} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} \quad \text{en možni izbor}$$

V tem primeru sta orientacija in
parametrizacija usklajeni:

$(\Gamma, \vec{T})$ ali $\vec{T}$

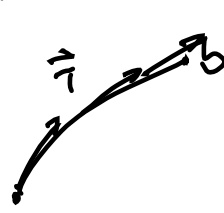
Vsaka krivulja je orientabilna

Odsekoma gladka krivulje



$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_n$$

Γ z robnimi točkama a, b

 (krivulja z robom $\partial\Gamma = \{a, b\}$)
Naj bo $\vec{T}$ orientacija Γ

Orientacija Γ porodi orientacijo $\partial\Gamma$

Ene od teh dveh točk je prva oz. začetna $\Gamma(Z_\Gamma)$ in druga je končna točka $\Gamma(K_\Gamma)$

Glede na sliko je $a = Z_\Gamma$ $b = K_\Gamma$

S tem je rob Γ orientiran skladno z orientacijo Γ

Naj bo Γ odsekoma gladka krivulja.

$$\Gamma = \Gamma_1 \cup \dots \cup \Gamma_n$$

1) Γ_j je gladka krivulja z robom

$$2) \Gamma_j \cap \Gamma_l = \emptyset \Leftrightarrow |j-l| \bmod n \neq 0$$

$$3) \Gamma_j \cap \Gamma_{j+1} = \{P_j\} \quad j \in \{1, \dots, n-1\} \quad \begin{array}{l} P_j \text{ robna} \\ \text{točka} \\ \text{ena sama točka z } \Gamma_j \text{ in } \Gamma_{j+1} \end{array}$$

Γ_n in Γ_1 se lahko sekata ali pa ne

Orientacija Γ je tak izbor orientacij

$$\Gamma_1, \dots, \Gamma_n, \text{ da je } K_{\Gamma_j} = P_j = Z_{\Gamma_{j+1}}$$

$$\text{če se } \Gamma_n \text{ in } \Gamma_1 \text{ sekata je } K_{\Gamma_n} = Z_{\Gamma_1}$$

Krivuljni integral

Dve vrsti krivuljnih integralov:

1) Integral skalarnege polja po krivulji Γ
(Orientacije tu ne potrebujemo)

Γ gladke krivulja omejena

$u: \Gamma \rightarrow \mathbb{R}$ zveza skalarne polje

$\vec{r}: [\alpha, \beta] \rightarrow \Gamma$ regularna parametrizacija

Definiramo $\int_{\Gamma} u ds := \int_{\alpha}^{\beta} u(\vec{r}(t)) \|\vec{r}'(t)\| dt$

Opombe:

1) $u=1 \Rightarrow$ je to dolžina Γ

2) Ta vrednost je neodvisna od
reg. parametrizacije

3) $u =$ dolžinska gostota $\Rightarrow$
dobimo maso Γ

Če je $u = \rho$ konstante, je Γ homogena

Če je $\Gamma_1, \dots, \cup \Gamma_n$ odsekom gladke

$$\text{je } \int_{\Gamma} u ds = \sum_{i=1}^n \int_{\Gamma_i} u ds$$

2) Krivuljni integral vektorskega polja
po orientirani krivulji

$\gg \gg$

Zadani: Γ homogena polkrožnica

$$\Gamma = \{ (x, y) \in \mathbb{R}^2, x^2 + y^2 = a^2, y \geq 0 \}$$

Leži težnica: ?

$$m(\Gamma) = \int_{\Gamma} \rho_0 ds$$

$$x_T(\Gamma) = \frac{1}{m(\Gamma)} \int_{\Gamma} x \rho_0 ds$$

$$y_T(\Gamma) = \frac{1}{m(\Gamma)} \int_{\Gamma} y \rho_0 ds$$

$$m(\Gamma) = \int_{\Gamma} \rho_0 ds = \pi a \rho_0$$

Parametrizacija:

$$x(t) = a \cos t \quad y(t) = a \sin t$$

$$ds = |\dot{\vec{r}}| dt$$

$$\dot{x} = -a \sin t$$

$$\dot{y} = a \cos t$$

$$x^2 + y^2 = a^2$$

$$m(\Gamma) = \int_0^{\pi} \rho_0 a dt = \rho_0 a \pi$$

$$x_T(\Gamma) = \frac{1}{m(\Gamma)} \int_0^{\pi} x \rho_0 a dt = 0$$

$$\frac{\rho_0 a \pi}{\rho_0 a \pi} \int_0^{\pi} \sin t a dt = \frac{a}{\pi} (-\cos t) \Big|_0^{\pi} = \frac{2}{\pi} \cdot a$$

Vztrajnostni moment okoli: x in y osi

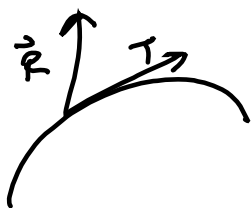
$$J_y = \int_{\Gamma} (x^2 + z^2) \rho ds = \int_0^{\pi} a^2 \cdot x^2 + a^2 \cdot dt = \rho_0 a^3 \int_0^{\pi} \frac{1 + \cos(2t)}{2} dt$$

$$= \frac{\pi}{2} \rho_0 a^3 = \frac{1}{2} m(\Gamma) \cdot a^2$$

2) Krivuljni integral vektorskega polja po orientirani krivulji
(Delo sile vzdolž $\vec{T}$)

$$\vec{R}: \vec{r} \rightarrow \mathbb{R}^3 \text{ vekt. polje}$$

$$\vec{R} = (x, y, z)$$



če je Γ gladke in

$\vec{r}: [\alpha, \beta] \rightarrow \Gamma$ param. ki je vsklajena z orientacijo Γ

$$\vec{T} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$$

Definiramo $\int_{\Gamma} \vec{R} d\vec{r} = \int_{\alpha}^{\beta} \vec{R}(\vec{r}(t)) \cdot \dot{\vec{r}}(t) dt$ ↙ skalarni produkt

$$= \int_{\alpha}^{\beta} \vec{R}(\vec{r}(t)) \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} \cdot \dot{\vec{r}} dt = \int_{\Gamma} (\vec{R} \cdot \vec{T}) ds$$

Rezultat je neodvisen od parametrizacije Γ , ki je vsklajena z orientacijo

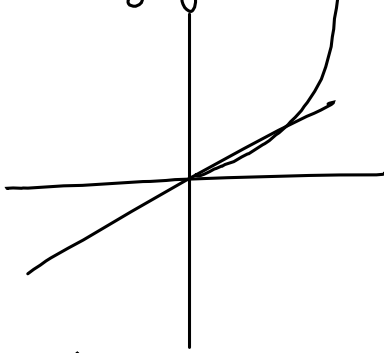
$$\int_{-\vec{\Gamma}} \vec{R} d\vec{r} = - \int_{\vec{\Gamma}} \vec{R} d\vec{r}$$

Začel:

$$\vec{P}(x, y, z) = (xy, z, x - z)$$

$$r(t) = (t, t, \frac{1}{2}t^2) \quad t \in [0, 1]$$

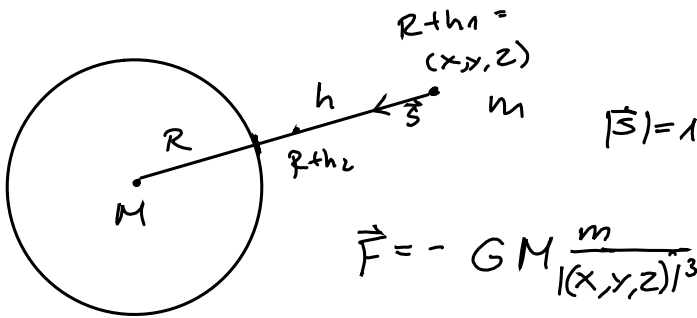
orientacija je ista ko je uskladjena s $\vec{r}$



$$\int_{\vec{r}} \vec{P} d\vec{r} = \int_0^1 (t^2, \frac{1}{2}t^2, t - \frac{1}{2}t^2) \cdot (1, 1, t) dt =$$

$$= \int_0^1 (\frac{3}{2}t^2 + t^2 - \frac{1}{2}t^2) dt = \frac{5}{6} - \frac{1}{8} = \frac{17}{24}$$

Zapęd:



$$\vec{F} = -GM \frac{m}{|(\vec{x}, \vec{y}, \vec{z})|^3} (\vec{x}, \vec{y}, \vec{z}) =$$

$$= -GM \frac{m}{r^3} \cdot \vec{r}$$

$\vec{r} \rightarrow \vec{S} \cdot \vec{r}$ + case od $R+h_1$ do $R+h_2$

$$\int_{\vec{r}} \vec{F} d\vec{r} = \int_{R+h_1}^{R+h_2} -GM \frac{m}{r^3} (\vec{r} \cdot \vec{S}) \cdot \vec{S} dt =$$

$$= -GMm \int_{R+h_1}^{R+h_2} \frac{dt}{t^2} = -GMm \left. \frac{1}{t} \right|_{R+h_1}^{R+h_2} = -GMm \left(\frac{1}{R+h_2} - \frac{1}{R+h_1} \right)$$

$$= \frac{GM(h_1 - h_2)m}{(R+h_1)(R+h_2)} \doteq \underbrace{\frac{GM}{R^2}}_g m \cdot \Delta h = m g \Delta h$$

Opomba

$$.) \int_{\vec{\Gamma}} \vec{R} d\vec{r} = \int_{\vec{\Gamma}} \underbrace{X dx + Y dy + Z dz}$$

diferencialna forma

..) če je Γ sklenjena ($Z_{\Gamma} = K_{\Gamma}$)

$$\oint_{\vec{\Gamma}} \vec{R} d\vec{r}$$

Naj bo $\vec{R}$ potencialno vektorsko polje

Naj bo Γ C^1 krivulja s parametrizacijo $\vec{r}(t)$

Orientacija usklajena z $\vec{r}$ $\vec{T} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$

$$\begin{aligned}\int_{\vec{\Gamma}} \vec{R} \, d\vec{r} &= \int_{\vec{\Gamma}} \text{grad} u \, d\vec{r} = \int_{\alpha}^{\beta} (u_x, u_y, u_z)(\vec{r}(t)) \cdot \dot{\vec{r}}(t) \, dt = \\&= \int_{\alpha}^{\beta} \dot{\vec{r}}(t) = (x(t), y(t), z(t)) = \int_{\alpha}^{\beta} (u_x \cdot \dot{x} + u_y \cdot \dot{y} + u_z \cdot \dot{z}) \, dt = \\&\quad \underbrace{\frac{d}{dt} (u(x(t), y(t), z(t)))}_{\text{chain rule}}\end{aligned}$$

$$= u(\vec{r}(\beta)) - u(\vec{r}(\alpha)) = u(\vec{r}_{\beta}) - u(\vec{r}_{\alpha})$$

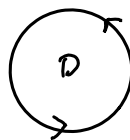
Trditev: Če je vek. polje $\vec{R}$ potencialno, je vrednost integrala $\vec{R}$ po orientirani krivulji $\vec{r}$ enaka razliki potenciala polja $\vec{R}$ med končno točko $\vec{r}$ in začetno točko $\vec{r}$

Posledica: Če je $\vec{R}$ potencialno polje in je Γ sklenjena je integral $\oint_{\Gamma} \vec{R} d\vec{r} = 0$

Zgled:

$$\vec{R} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$$

$$\Gamma = \{ (x, y, 0); x^2 + y^2 = R^2; R > 0 \}$$



$$x = R \cos t \quad t \in [0, 2\pi)$$

$$y = R \sin t$$

$$z = 0$$

$$\dot{x} = -R \sin t$$

$$\dot{y} = R \cos t$$

$$\dot{z} = 0$$

$$\begin{aligned} \int_{\Gamma} \vec{R} dr &= \int_0^{2\pi} \left(\frac{1}{R^2} R \sin t, \frac{1}{R^2} R \cos t, 0 \right) \cdot (-R \sin t, R \cos t, 0) dt = \\ &= \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = 2\pi \neq 0 \end{aligned}$$

$\Rightarrow$ To polje ni potencialno v $\mathbb{R}^3 - \{z=0\}$

(Vemo pa da je $\text{rot } \vec{R} = \vec{0}$)

Izrek (karakterizacija potencialnih vek. polj)

Naj bo $D \subseteq \mathbb{R}^3$ odp in $\vec{R}: D \rightarrow \mathbb{R}^3$ zvezo vektorsko polje:

Naslednje tri izjave so ekvivalentne:

1) $\vec{R}$ je potencialno na D

$$(\exists U \in C^1(D), \vec{R} = \text{grad } U)$$

2) Integral $\vec{R}$ po orientiranih krivuljah je neodvisen od krivulj z isto začetno in isto končno točko

3) Integral po sklenjenih krivuljah je enak 0

Nečrto Level: žile, vaskularno
Raban užitni barva rgb $\rightarrow$ path
Ustnik black $\rightarrow$ path vrb ml
to dva vrednoti skeniranje!
Trat

Dokaz:

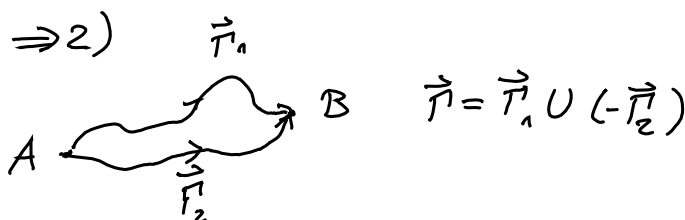
1) $\Rightarrow$ 3)

$\vec{r}$ sklenjena: $t \rightarrow \vec{r}(t)$ parametrizacija C'
 $t \in [\alpha, \beta]$; $\vec{r}(\alpha) = \vec{r}(0)$

$$\int_{\vec{r}} \vec{R} d\vec{r} = \int_{\alpha}^{\beta} \vec{R}(\vec{r}(t)) \dot{\vec{r}}(t) dt = \vec{R}(\vec{r}(\beta)) - \vec{R}(\vec{r}(\alpha)) = 0$$

$$\propto \frac{d}{dt} (\vec{R}(\vec{r}(t))) dt$$

3) $\Rightarrow$ 2)



$$\int_{\vec{r}} \vec{R} d\vec{r} = 0$$

$$\int_{\vec{r}_1} \vec{R} d\vec{r} - \int_{\vec{r}_2} \vec{R} d\vec{r} = 0 \rightarrow \int_{\vec{r}_1} \vec{R} d\vec{r} = \int_{\vec{r}_2} \vec{R} d\vec{r}$$

2) $\Rightarrow$ 1)

Potrebujemo potencial u . $\vec{\nabla} u = \vec{R}$

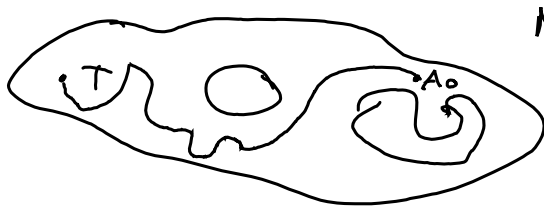
Naj bo D povezana (če ni, delamo po vsaki komponenti posebi)

D je s potmi povezana

Za vsako dve točki obstaja odsekoma

gladka pot od A do B

Naj bo $A_0 \in D$ neka točka



Naj bo $T \in D$.

Naj bo $\vec{r} \in D$

odsekoma gladka pot od A_0 do T

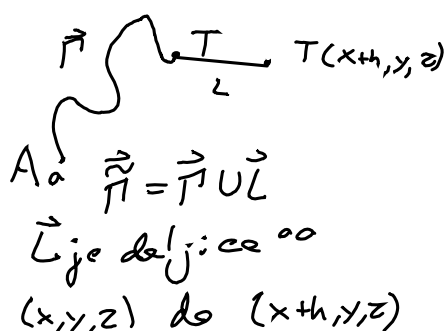
$$\text{Definiramo } u(T) = u(x, y, z) = \int_{\vec{r}} \vec{R} d\vec{r}$$

le dobro definirano, ker ni odvisen od poti od A_0 do T

$$\text{Trdimo: } \vec{\nabla} u = (u_x, u_y, u_z) = \vec{R} = (X, Y, Z)$$

$$\lim_{h \rightarrow 0} \frac{u(x+h, y, z) - u(x, y, z)}{h} =$$

$$u(x, y, z) = \int_{\vec{r}} \vec{R} d\vec{r}$$



$\vec{L}$ je deljice od

(x, y, z) do $(x+h, y, z)$

$$\frac{u(x+h, y, z) - u(x, y, z)}{h} = \frac{1}{h} \left(\int_{\vec{r} \cup \vec{L}} \vec{R} d\vec{r} - \int_{\vec{r}} \vec{R} d\vec{r} \right) =$$

$$= \frac{1}{h} \int_{\vec{L}} \vec{R} d\vec{r}$$

parametrizacija $\vec{L}: \vec{r}: t \rightarrow (x+h, y, z)$

$$t \in [0, 1]$$

$$\vec{r}(t) = (x+h, y, z) = h(1, 0, 0)$$

$$\frac{1}{h} \int_0^1 \vec{R}(x+h, y, z) h(1, 0, 0) dt =$$

$$= \int_0^1 X(x+h, y, z) dt \xrightarrow{h \rightarrow 0} X(x, y, z)$$

Res $u_x = X$ podobno se u_y in u_z

Opombe:

$$1) \int_{\vec{r}} \vec{\nabla} u dr = u(k_F) - u(2T_F)$$

če je v nek drug potencial $\vec{R} = -\vec{\nabla} v = \vec{\nabla} u$

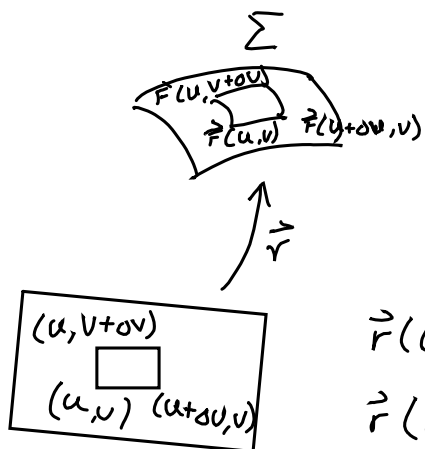
$$\mathcal{D} \text{ povezane} \Rightarrow v = u + C$$

2) $\mathcal{D}$ zvezdasto glede na neko točko, $\text{rot } \vec{R} = \vec{0}$

$$u(T) = u(x, y, z) \int_{[A_0, T]} \vec{R} d\vec{r} = \int_0^1 (x X(t_x, t_y, t_z) + \\ y Y(x, y, z) + z Z(x, y, z)) dt$$

Površine ploške

$\Sigma \in \mathbb{R}^3$ gladka ploška



$$\vec{r}(u+\Delta u, v) - \vec{r}(u, v) \approx \vec{r}_u(u, v) \Delta u$$

$$\vec{r}(u, v+\Delta v) - \vec{r}(u, v) \approx \vec{r}_v(u, v) \Delta v$$

$$\Delta P = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

Definiramo:

$$P(\Sigma) = \iint_D |\vec{r}_u \times \vec{r}_v| du dv$$

Opomba: $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$

$$(\vec{r}_u \times \vec{r}_v) \cdot (\vec{r}_u \times \vec{r}_v) = |\vec{r}_u|^2 |\vec{r}_v|^2 - (\vec{r}_u \cdot \vec{r}_v)^2 \geq 0$$

$$E = |\vec{r}_u|^2$$

$$F = \vec{r}_u \cdot \vec{r}_v$$

$$G = |\vec{r}_v|^2$$

$$= EG - F^2 = \begin{vmatrix} E & F \\ F & G \end{vmatrix}$$

$$P(\Sigma) = \iint \sqrt{EG - F^2} du dv$$

Paršine graf

$$\Sigma = \{(x, y, f(x, y)) \mid (x, y) \in D \subseteq \mathbb{R}^2\}$$

$$\vec{r}(x, y) = (x, y, f(x, y))$$

$$E = 1 + f_x^2$$

$$\vec{r}_x = (1, 0, f_x(x, y))$$

$$F = f_x f_y$$

$$\vec{r}_y = (0, 1, f_y(x, y))$$

$$G = 1 + f_y^2$$

$$EG - F^2 = (1 + f_x^2)(1 + f_y^2) - f_x^2 f_y^2 =$$

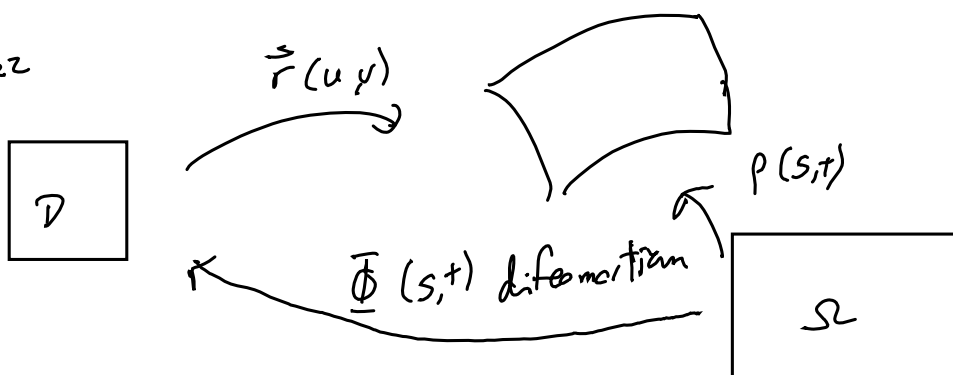
$$= 1 + f_x^2 + f_y^2$$

$$P(Q(f)) = \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dx, dy$$

Opomba: E, F, G so koeficienti I Fundamentalne forme plošče $Eu^2 + 2Fuv + Gv^2$

Trditev: Definicija površne plošče je neodvisna od regularne parametrizacije plošče

Dokaz



$$\vec{P}(s,t) = (\vec{r} \circ \underline{\Phi})(s,t)$$

$$\iint_{\Omega} |\vec{P}_s \times \vec{P}_t| ds dt = \iint (\vec{r}_u \times \vec{r}_v) du dv$$

$$\underline{\Phi}: \Omega_{(s,t)} \longrightarrow P_{(u,v)}$$

$$\underline{\Phi}(st) = (U(st), V(st))$$

$$P(s,t) = \vec{r}(U(st), V(st))$$

$$P_s = \vec{r}_u \cdot U_s + \vec{r}_v \cdot V_s$$

$$P_t = \vec{r}_u \cdot V_t + \vec{r}_v \cdot V_t$$

$$P_s \cdot P_t = (\vec{r}_u \times \vec{r}_v) U_s V_t - (\vec{r}_u \times \vec{r}_v) U_t V_s =$$

$$= (\vec{r}_u \times \vec{r}_v) (U_s V_t - U_t V_s) = \dots \begin{vmatrix} U_s & U_t \\ V_s & V_t \end{vmatrix}$$

$$|P_s \times P_t| = |\vec{r}_u \times \vec{r}_v| |J\underline{\Phi}|$$

$$\iint_D |\vec{r}_u \times \vec{r}_v| du dv = \iint_{\Omega} |\vec{r}_u \times \vec{r}_v| \underline{\Phi}(st) |J\underline{\Phi}(st)| ds dt =$$

(u,v) = \underline{\Phi}(s,t)

$$= \iint_{\Omega} |P_s \times P_t| ds dt$$

Zugel:

$$1) S(0, a) = \{ (x, y, z) \mid x^2 + y^2 + z^2 = a^2 \}$$

$$x(\vartheta, \rho) = a \cos \vartheta \sin \rho$$

$$y(\vartheta, \rho) = a \cos \rho \sin \vartheta$$

$$z(\vartheta, \rho) = a \cos \vartheta$$

$$\vec{r}(\vartheta, \rho) = x(\vartheta, \rho), y(\vartheta, \rho), z(\vartheta, \rho)$$

$$\vec{r}_{\vartheta} = (a \cos \rho \cos \vartheta, a \sin \rho \cos \vartheta, -a \sin \vartheta)$$

$$\vec{r}_{\rho} = (-a \sin \rho \sin \vartheta, a \cos \rho \sin \vartheta, 0)$$

$$E = a^2$$

$$F = 0$$

$$P(S) = \int_0^{4\pi} \int_0^{\pi} a^2 \sin \vartheta \, d\vartheta \, d\rho =$$

$$G = a^2 \sin^2 \vartheta = a^2 2\pi (-\cos \vartheta) \Big|_0^{\pi} = 4\pi a^2$$

$$2) f(x, y) = \frac{1}{2} (x^2 + y^2)$$

$$D = \{ (x, y) \mid x^2 + y^2 \leq 1 \}$$

$$f_x = x$$

$$x = \cos \varphi$$

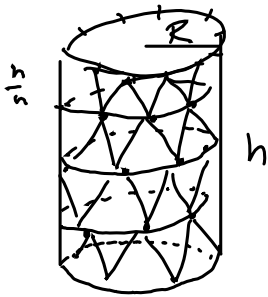
$$f_y = y$$

$$y = \sin \varphi$$

$$P(G(f)) = \iint_D \sqrt{1 + x^2 + y^2} \, dx, dy =$$

$$\int_0^1 \int_0^{2\pi} r \sqrt{1 + r^2} \, d\rho \, dr = 2\pi \frac{2}{3} (1 + r^2)^{\frac{3}{2}} \frac{1}{2} \Big|_0^1 = \frac{2\pi}{3} (2^{\frac{3}{2}} - 1)$$

zglede (Schwarze Lankerne at. Schwarze Skorn

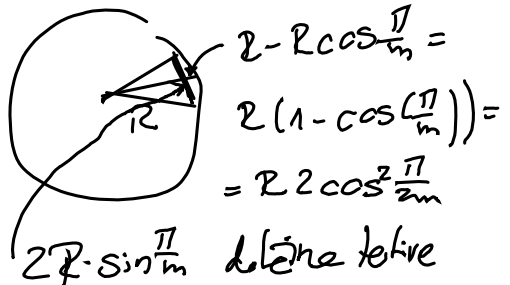
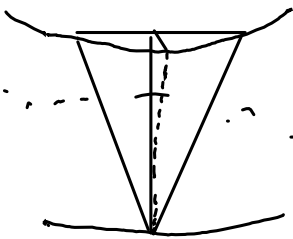


Prizajemo površino $2\pi R h$

m lokov evoli dolžil

n evoli trakov

$2nm$ trikotnikov (ne ležijo na ploskvi)



Ploščina trikotnika:

$$\frac{1}{2} R \sin \frac{\pi}{n} \sqrt{\frac{h^2}{n^2} + R^2 \sin^2 \frac{\pi}{2n}}$$

Površina vseh teh trikotnikov je

$$P(n, m) = 2nm R \sin \frac{\pi}{m} \sqrt{\frac{h^2}{n^2} + 4 R^2 \sin^2 \frac{\pi}{2m}}$$

$$1) \lim_{m \rightarrow \infty} P(m, m) = \lim_{m \rightarrow \infty} 2R m^2 \sin \frac{\pi}{m} \sqrt{\frac{h^2}{m^2} + 4 R^2 \sin^2 \frac{\pi}{2m}} =$$

$$= 2R \lim_{m \rightarrow \infty} m^2 \frac{\pi}{m} \sqrt{\frac{h^2}{m^2} + 4 R^2 \frac{\pi^4}{16 m^4}} =$$

$$= 2R \lim_{m \rightarrow \infty} \pi \sqrt{h^2 + \frac{R^2 \pi^4}{4 m^2}} = 2\pi R h$$

$$\lim_{m \rightarrow \infty} P(m^2, m) = 2R \lim_{m \rightarrow \infty} m^3 \frac{\pi}{m} \sqrt{\frac{h^2}{m^4} + \frac{4 R^2 \pi^4}{16 m^4}} =$$

$$= 2R \pi \lim_{m \rightarrow \infty} \sqrt{h^2 + \frac{\pi^4 R^2}{4}} > 2\pi R h$$

$$3) \lim_{m \rightarrow \infty} P(m^3, m) = 2R \lim_{m \rightarrow \infty} m^4 \frac{\pi}{m} \sqrt{\frac{h^2}{m^6} + \frac{4 R^2 \pi^4}{16 m^4}} =$$

$$= 2R \lim_{m \rightarrow \infty} \sqrt{h^2 + \frac{\pi^4 R^2}{4}} m^2 = \infty$$

Orientacija ploskev

(za krivulje :
 $|\vec{T}|=1$ kjeri tangentni vektorji zadovoljijo Γ)

$\Sigma \subseteq \mathbb{R}^3$ gladka ploskev

orientacija Σ je vezan izbor enotske normale na Σ

$$\vec{N} : \Sigma \rightarrow \mathbb{R}^3$$

$$|\vec{N}|=1 \quad \vec{N}(x,y,z) \perp T_{(x,y,z)}\Sigma$$

če Σ lahko orientiramo je Σ **orientabilna**
(sicer je **neorientabilna**)

$(\Sigma, \vec{N})$ če je Σ povezana in orientabilna, ima natančno dve različni orientaciji

$$(\Sigma, \vec{N}), (\Sigma, -\vec{N})$$

Zgledi:

1) $\Sigma = G(p) = \{(x, y, f(x, y)), (x, y) \in D \subseteq \mathbb{R}^n\}$

$$f \in (C_1(D))$$

$$\vec{N} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}$$

2) $S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = R^2\}$

$$\vec{N} = \frac{(x, y, z)}{R}$$

zunanja normala:
 $x^2 + y^2 + z^2 = R^2 \quad R > 0$



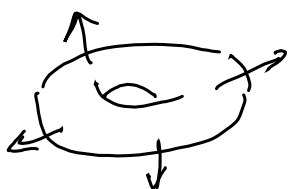
3)



$$\Sigma = \{(x, y, z) ; x^2 + y^2 = R^2 ; z \in \mathbb{R}\}$$

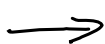
$$\vec{N} = \frac{(x, y, 0)}{R}$$

4)



v $\mathbb{R}^3$ so vse sklenjene (kompaktno) ploskve orientabilne

5)



Möbiusov trak



N: orientabilne

$\Sigma \subseteq \mathbb{R}^3$ je lahko orientabilna ali pa ni orientabilna

2 možni orientaciji:
za prazen Σ

6) $\Sigma \in \mathbb{R}^2 \times \{0\}$

če ni rečeno drugače je orientirano z
normalo $(0,0,1) = \vec{n}$
(pozitivna orientacija)

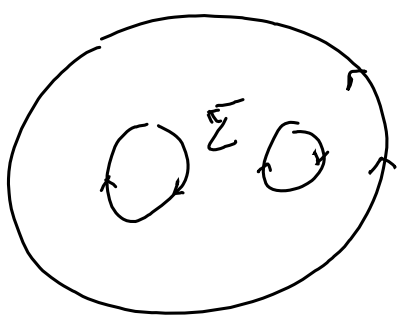
Naj bo Σ gladka ploskev z robom (končna
unija odsekov gladkih sklenjenih krivulj)

Naj bo $(\Sigma, \vec{N})$ ploskev z orientacijo

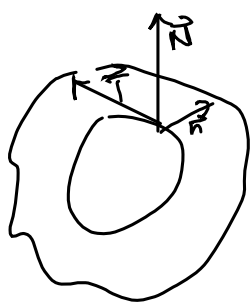
Skladna oz. koherentna orientacija $\partial\Sigma$ je

definirana takole: če sprehajalec stoji

na robu Σ in ima glavo
v smeri normale, ^(stoji kot normala) je rob
 $\partial\Sigma$ orientiran tako da Σ
leži na levi strani sprehajalca,
če le tako gleda v smeri normale



če $\bar{\Sigma} \subseteq \tilde{\Sigma}$



$$\vec{T} = \vec{N} \times \vec{n}$$

$\vec{n}$... zunanja normala
(na krivuljo)

$$(\Sigma, \vec{N}) \rightsquigarrow (\partial\Sigma, \vec{T})$$

skladna orientacija

$$(\Sigma, -\vec{N}) \rightsquigarrow (\partial\Sigma, -\vec{T})$$

Opomba: $\mathbb{R}^2$

$\tilde{c}: \tilde{D} \rightarrow \Sigma$ regularna parametrizacija

$$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$$

Orientacija $\vec{N}$ je vključena s parametrizacijo $\tilde{r}$

Odslekovno gladke ploskev

$$\Sigma = \Sigma_1 \cup \dots \cup \Sigma_n \subset \mathbb{R}^n$$

1) $\Sigma_1, \dots, \Sigma_n$ so gladke ploskve z robom

2) če $\Sigma_i \cap \Sigma_j \neq \emptyset$ je presekok del roba Σ_i in Σ_j
 $i \neq j$

3) $\Sigma_i \cap \Sigma_j \cap \Sigma_k \in \{\emptyset, \{x\}\}$
 $i \neq j \neq k$ (presekok je ali prazen ali točka)

Orientacija $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_n$ če obstaja,
je tak nabor orientacij, da sta na vseh
presekih

$\Pi_{ij} = \Sigma_i \cap \Sigma_j$ inducirani orientaciji:
nasprotni

Ploskovni integral

1) Ploskovni integral skalarnega polja

$$u: \Sigma \rightarrow \mathbb{R} \quad \text{vezno skalarno polje}$$

= regularna parametrizacija

$$\vec{r}: D \rightarrow \Sigma$$

$$\iint_{\Sigma} u dS := \iint_D u(\vec{r}(u,v)) \sqrt{EG-F^2} \, du dv =$$

$$\iint_D u(\vec{r}(u,v)) \|\vec{r}_u \times \vec{r}_v\| \, du dv$$

$$E = |\vec{r}_u|^2 \quad F = \vec{r}_u \cdot \vec{r}_v \quad G = |\vec{r}_v|^2$$

Opomba: $u=1 \Rightarrow$ to je površna plošče

Opomba: Ta vrednost je neodvisna od regularne parametrizacije $\vec{r}$

Opomba: Tu ne potrebujemo orientacije

Ta integral obstaja tudi na Möbiusovem traku

Opomba: $u=\rho \dots$ površinska gostota, kateri je

$$\iint_{\Sigma} \rho dS = m(\Sigma)$$

ρ konstanten $\Rightarrow$ plošče je homogena

zged

$$S = \{ (x, y, z) ; x^2 + y^2 + z^2 = R^2 \} \quad R > 0$$

homogene Sphäre

$$J_z = \iint_S \rho (x^2 + y^2) dS$$

$$(\vartheta, \rho) \mapsto (R \cos \vartheta \sin \varphi, R \sin \vartheta \cos \varphi, R \cos \vartheta)$$

$$\sqrt{EG - F^2} = R^2 \sin \vartheta$$

$$J_z = \int_0^{2\pi} \int_0^\pi \rho_0 R^2 \sin^2 \vartheta R^2 \sin \vartheta d\vartheta d\varphi =$$

$$= \rho_0 2\pi R^4 \int_0^\pi \sin^3 \vartheta d\vartheta = 2\pi \rho_0 R^4 \int_{-1}^1 (1-t^2) dt$$
$$2\pi \rho_0 R^4 \cdot 2 \left(1 - \frac{1}{3}\right) = \frac{8}{3} \rho_0 R^4 =$$

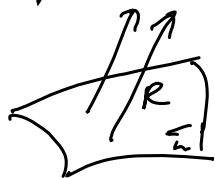
$$m(S) = 4\pi R^2 \rho_0$$

$$= \frac{2}{3} m R^2$$

2) Ploskovni integral vektorskega polja po orientirani ploskvi

$(\Sigma, \vec{N}) \quad \vec{R}: \Sigma \rightarrow \mathbb{R}^3$ veko vekt. polje

$$\int_{\vec{R}} \vec{R} d\vec{r} = \int_{\Gamma} (\vec{R} \cdot \vec{T}) ds$$



Pretok $\vec{R}$ skozi Σ

$$\iint_{\Sigma} \vec{R} d\vec{S} = \iint_{\Sigma} (\vec{R} \cdot \vec{N}) dS \quad \text{Neodvisno reg. parametrizacije, usklajeno z orientacijo } \Sigma$$

$$\text{Tudi } \iint_{\Sigma} \vec{R} (-\vec{N}) dS = - \iint_{\Sigma} (\vec{R} \cdot \vec{N}) dS$$

$$\vec{N} \text{ je ustrezno ko } \vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

Polem:

$$\iint_{\Sigma} \vec{R} d\vec{S} = \iint_{\Sigma} \vec{R} \cdot \vec{N} dS = \iint_{\mathcal{D}} \vec{R}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) du dv$$

$$= \iint_{\mathcal{D}} (\vec{r}_u, \vec{r}_v, \vec{R}) du dv$$

$\mathcal{D}$ $\wedge$ mešani produkt

če je $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_n$

$$\iint_{\Sigma} u dS = \sum_{j=1}^n \iint_{\Sigma_j} u dS$$

$$\iint_{\Sigma} \vec{R} d\vec{S} = \sum_j \iint_{\Sigma_j} \vec{R} d\vec{S}$$

2. gelöst:

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = R^2\}$$

$$\vec{r} = \frac{\vec{r}}{|\vec{r}|^3} = \left(\frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right)$$

$$= -\vec{\nabla} \frac{1}{|\vec{r}|}, \quad \operatorname{div} \vec{r} = 0 \quad \text{da } \vec{r} = 0 \text{ in } \mathbb{R}^3 \setminus \{0\}$$

$$\vec{N} = (0, 0, 1) = (0, 0, 1)$$

$$\iint_S \vec{r} d\vec{s} = \iint_S \frac{\vec{r}}{|\vec{r}|^3} \cdot \frac{\vec{r}}{|\vec{r}|} d\vec{s} = \iint_S \frac{R^2}{R^4} d\vec{s} =$$

$$= \iint_S \frac{1}{R^2} d\vec{s} = \frac{1}{R^2} 4\pi R^2 = 4\pi$$

Opombe

Diferencialna 1-forma

$$\int_{\vec{r}} \vec{R} d\vec{r} = \int_{\vec{r}} X dx + Y dy + Z dz$$

$$\iint \vec{R} d\vec{S} = \int X dx \wedge dz + Y dz \wedge dx + dz \wedge dy$$

Integralni izreki

$$\int_a^b f(x) dx = f(b) - f(a)$$

$$\Gamma = [a, b]$$

$$\vec{T} = \vec{1}$$

Integral po orientiranem intervalu I po

$\delta[a, b]$ orientiranem skupju s orientirjo $[a, b]$

Že vemo:

$$\int_{\Gamma} \vec{\nabla} u d\vec{r} = u(KT_{\Gamma}) - u(AT_{\Gamma})$$

oblike integralnih izrekov

M orientirana mnogokotna z robom ∂M

$$M=[a,b] \quad M=\pi, \quad M=D \subset \mathbb{R}^2, \quad M=\Sigma \subset \mathbb{R}^3, \quad M=D \subset \mathbb{R}^3$$

Izrek: imajo oblike:

$$\int_{\partial M} \omega = \int_M d\omega \quad (\text{Stokesov izrek})$$

↑
reč objekt

Gaussov izrek (Gaussov-ostrogredski izrek)

Naj bo $\Omega \subseteq \mathbb{R}^3$ omejena odprta množica z robom sestavljenim iz končnega števila odsekov gladkih sklenjenih ploskev, orientiranih z zunanjo normalo glede na Ω



Naj bo $\vec{R}: \Omega \rightarrow \mathbb{R}^3 \subset \mathbb{R}^n$ vekt. polje

Tedaj je

$$\iint_{\partial \Omega} \vec{R} d\vec{S} = \iiint_{\Omega} \operatorname{div} \vec{R} dV$$

???

preveri
resilnost
tega

$$\iiint_D \operatorname{div} \vec{R} dV = \iint_{\partial D} \vec{R} d\vec{S} = \iint_{\partial D} \vec{R} \cdot \vec{n} dS$$

$n=2$: $D \subseteq \mathbb{R}^2$

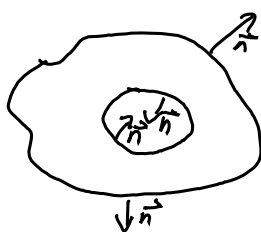
omejena odprta množica v $\mathbb{R}^2$ z odsekom gladkim robom sestavljenim iz končnega števila odsekov gladkih sklenjenih krivulj

Naj bo $\vec{R}: D \rightarrow \mathbb{R}^2$ vekt. polje

Tedaj je

$$\iint_D \operatorname{div} \vec{R} dS = \int_{\partial D} \vec{R} \cdot \vec{n} dS$$

← zunanja enotska normala



Greenov izrek (greenova formula):

Naj bo $D \subseteq \mathbb{R}^2$ omejena, odp. z odsekoma gladkim robom sestavljenim iz končnega števila odsekov gladkih sklenjenih krivulj, orientiranih pozitivno glede na D

Naj bosta $x, y \in C^1(\bar{D})$

Torej

$$\int_{\partial D} x dx + y dy = \iint_D (y_x - x_y) dx dy$$

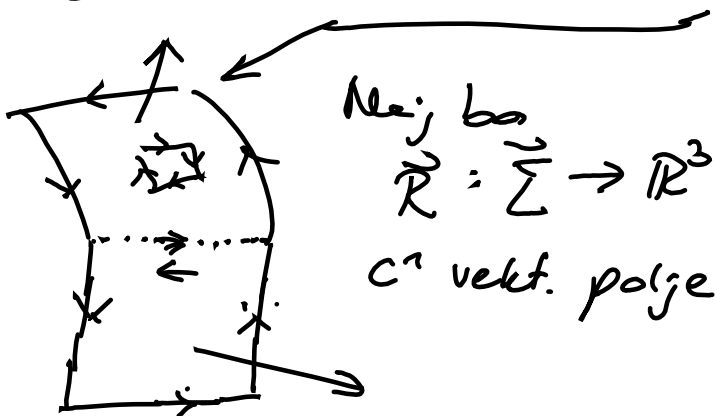
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$$\int_{\partial D} \vec{R} d\vec{r}$$

$$\vec{R} = (x, y)$$

Stokesov izrek:

Naj bo $\Sigma \subset \mathbb{R}^3$ omejena, orientirana, odsekana
gladka ploskev z odsekoma gladkim
robom, sestavljenim iz končnega
števila odsekov gladkih sklenjenih
krivulj orientiranih skladno z orientacijo Σ



Naj bo
 $\vec{R} : \Sigma \rightarrow \mathbb{R}^3$
 C^1 vekt. polje

Torej je

$$\int_{\partial \Sigma} \vec{R} d\vec{r} = \iint_{\Sigma} \text{rot} \vec{R} d\vec{S}$$

Zufall

$$1) D = K(0, R_0) \quad R_0 > 0$$

$$\vec{R} = (x, y, z)$$

$$X(x, y, z) = x$$

$$Y(x, y, z) = y$$

$$Z(x, y, z) = z$$

$$\iiint_D \operatorname{div} \vec{R} dV =$$

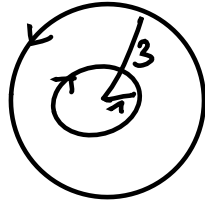
$$= \iiint_D 3 dV = 3V(D) = 3 \frac{4\pi R_0^3}{3} = 4\pi R_0^3$$

$$\iint_{\partial D} \vec{R} d\vec{s} = \iint_{S(0, R_0)} (x, y, z) \overbrace{\frac{(x, y, z)}{R_0}}^{\vec{n}} dS =$$

$$= R_0 \iint_{S(0, R_0)} 1 dS = R_0 (4\pi R_0^2)$$

$$2) D = \{ (x, y) ; 1 < x^2 + y^2 < 9 \}$$

$$X = -y \quad Y = x$$



$$\iint_D (X_y - Y_x) dx dy =$$

$$= \iint_D 2 dx dy = 2 \text{Pl}(D) =$$

$$2 (\pi \cdot 9 - \pi \cdot 1) = 16\pi$$

$$\int_D X dx + Y dy = \int_{x^2+y^2=3^2} -y dx + x dy + \int_{x^2+y^2=1} -y dx + x dy$$

$\text{Ⓔ} \quad \cdot) \quad \text{Ⓔ} \quad \cdot\cdot)$

$$\cdot) x = 3 \cos t$$

$$y = 3 \sin t \quad t \in [0, 2\pi]$$

$$dx = -3 \sin t dt$$

$$dy = 3 \cos t dt$$

$$\int_{x^2+y^2=3^2} -y dx + x dy = \int_0^{2\pi} (-3 \sin t) (-3 \sin t dt) +$$

$$+ 3 \cos t \cdot 3 \cos t dt =$$

$$= 9 \int_0^{2\pi} 1 dt = 18\pi$$

$$\cdot\cdot) x = \cos t \quad y = \sin t \quad t \in [0, 2\pi] \text{ de } 0$$

$$\int_{x^2+y^2=1} -y dx + x dy = \int_0^{2\pi} (-\sin t) (-\sin t dt) + \cos t \cdot \cos t dt =$$

$$= \int_0^{2\pi} 1 dt = 2\pi$$

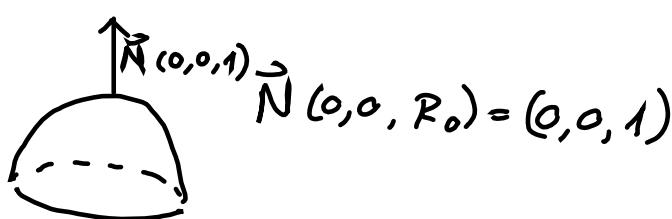
$$\cdot) + \cdot\cdot) = 16\pi$$

$D \subseteq \mathbb{R}^2$ let v greenough's formula.

$$\int_{\partial D} -y dx + x dy = 2 \cdot Pl(D)$$

$$Pl = \frac{1}{2} \int_{\partial D} -y dx + x dy$$

$$3) R_0 > 0 \quad \Sigma: \{x, y, z \in \mathbb{R}^3; x^2 + y^2 + z^2 = R_0^2, z > 0\}$$



$$\vec{r} = (x, y, x, x - y + z)$$

$$\int_{\partial \Sigma} \vec{r} d\vec{r}$$

parametrization $\partial \Sigma$

$$x = R_0 \cos t$$

$$y = R_0 \sin t$$

$$z = 0 \quad t \in [0, 2\pi)$$

$$\vec{r}(R_0 \cos t, R_0 \sin t, 0) =$$

$$= (0, R_0 \cos t, R_0 (\cos t - \sin t))$$

$$\dot{x} = -R_0 \sin t$$

$$\dot{y} = R_0 \cos t$$

$$\dot{z} = 0$$

$$\vec{r} \cdot \dot{\vec{r}} = R_0^2 \cos^2 t$$

$$\int_{\partial \Sigma} \vec{r} d\vec{r} = \int_{\partial \Sigma} x dx + y dy + z dz = \int_0^{2\pi} -R_0^2 \cos^2 t dt =$$

$$= -R_0^2 \int_0^{2\pi} \frac{1 + \cos(2t)}{2} dt = -\pi R_0^2$$

$$\text{rot } \vec{r} = (-1, y-1, 1-z)$$

$$\iint_{\Sigma} \text{rot } \vec{r} dS = \iint_{\Sigma} (1, y-1, 1-z) \cdot \frac{(x, y, z)}{R_0} dS$$

$$= \frac{1}{R_0} \iint_{\Sigma} (-x + y^2 - y + z - z^2) dS$$

Parametrization sphere

$$x = R_0 \cos \varphi \sin \vartheta$$

$$y = R_0 \sin \varphi \sin \vartheta$$

$$z = R_0 \cos \vartheta$$

$$\varphi \in [0, 2\pi)$$

$$\vartheta \in [0, \frac{\pi}{2}]$$

$$\sqrt{EG - F^2} = R_0^2 \sin \vartheta$$

$$I = \frac{1}{R_0} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} R_0^2 \sin \vartheta (-R_0 \cos \varphi \sin \vartheta + R_0^2 \sin^2 \vartheta \sin \vartheta - R_0 \sin \varphi \sin \vartheta + R_0 \cos \vartheta - R_0^2 \cos^2 \vartheta) d\vartheta d\varphi$$

$$= R_0 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} R_0^2 \frac{1 - \cos(2\varphi)}{2} \sin^3 \vartheta + R_0 \frac{1}{2} \sin(2\vartheta) - R_0^2 \cos^2 \vartheta \sin \vartheta d\vartheta d\varphi =$$

$$= R_0 \int_0^{2\pi} (\pi R_0 \sin^3 \vartheta + \pi R_0 \sin(2\vartheta) - 2\pi R_0^2 \cos^2 \vartheta \sin \vartheta) d\vartheta$$

$$= R_0^2 \pi \int_0^{\frac{\pi}{2}} R_0 (1 - \cos^2 \vartheta \sin \vartheta - 2R_0 \cos^2 \vartheta \sin \vartheta + \sin 2\vartheta) d\vartheta$$

$$= \pi R_0^2 \left(R_0 \int_0^{\frac{\pi}{2}} (1 - 3\cos^2 \vartheta) \sin \vartheta + \frac{1}{2} \cos(2\vartheta) \Big|_0^{\frac{\pi}{2}} \right)$$

$$= \pi R_0^2 \left(R_0 \int_0^1 (1 - 3t^2) dt + 1 \right)$$

$$\underbrace{R_0(1-1)}_{0} + 1$$

$$\leadsto = \pi R_0^2$$

$$\vec{R} = (2x, x, x-y+z)$$

$$\text{rot } \vec{R} = (-1, y-1, 1-z)$$



$$\Sigma U \{ (x, y, 0) \mid x^2 + y^2 \leq R^2 \}$$

Gauss'satz

$$\iint_D \text{rot } \vec{R} d\vec{S} = \iiint_D \text{div}(\text{rot } \vec{R}) dV$$

$$\iint_{\Sigma} \text{rot } \vec{R} d\vec{S} + \iint_{\Delta} \text{rot } \vec{R} d\vec{S}$$

$$\Delta: N = (0, 0, -1)$$

$$\iint_{\Delta} \text{rot } \vec{R} d\vec{S} = \iint_{\Delta} (-1, y-1, 1) (0, 0, -1) dS$$

$$= - \iint_{\Delta} 1 dS = -\pi R_0^2$$

Gaussov izrek v $\mathbb{R}^2$

$\Rightarrow$ Greenova formula

Dokt v Greenovi formuli:

V Gaussovem izreku za $n=2$

$$\int_{\partial D} X dx + Y dy = \int_{\partial D} \vec{P} \cdot d\vec{r} = \int_{\partial D} \vec{P} \cdot \vec{T} \cdot ds =$$

∂D je pozitivno orientiran

$$= \int_{\partial D} (X \cdot T_1 + Y T_2) ds \quad \vec{T} = (T_1, T_2)$$



$$\int_{\partial D} \vec{P} \cdot \vec{T} ds = \int_{\partial D} (-i) \vec{P} \cdot (-i \vec{T}) ds$$

$$\vec{P} = (x, y) \quad "-i \vec{P}" = (y, -x)$$

$$\vec{n} = (T_2, -T_1) = "T_1 + i T_2"$$

$$= \int_{\partial D} \underbrace{(y \cdot T_2 + (-x) \cdot (-T_1))}_{(y, -x) \cdot \vec{n}} ds$$

gauss

$$= \iint_D (y_x - x_y) dx dy$$

Stokesov izrek $\Rightarrow$ Greenova formula

$$\int_{\partial \Sigma} \vec{P} d\vec{r} = \iint_{\Sigma} \text{rot } \vec{P} d\vec{S}$$

$$\Sigma = D, \quad \vec{N} = (0, 0, 1)$$

$$\vec{P} = (X(x, y), Y(x, y), 0)$$

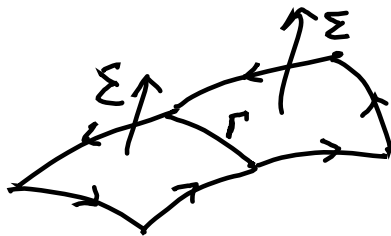
$$\text{rot } \vec{P} = (0, 0, Y_x - X_y)$$

$$\Rightarrow \int_{\partial D} \vec{P} d\vec{r} = \iint_D (Y_x - X_y) dx dy$$

$$\Sigma = \Sigma_1 \cup \Sigma_2 \subseteq \mathbb{R}^3$$

Plošče v $\mathbb{R}^3$, orientirane, omejene
z odsekoma gladkim robom

$$\overline{\Sigma}_1 \cap \overline{\Sigma}_2 = \Gamma \text{ krivulje}$$



$$\vec{R}: \overline{\Sigma} \rightarrow \mathbb{R}^3 \text{ } C^1 \text{ vekt. polje}$$

$$\iint_{\Sigma} \operatorname{rot} \vec{R} d\vec{S} = \iint_{\Sigma_1} \operatorname{rot} \vec{R} d\vec{S} + \iint_{\Sigma_2} \operatorname{rot} \vec{R} d\vec{S} =$$

problema: na ploščah 0

$$= \iint_{\partial \Sigma_1} \vec{R} d\vec{r} + \int_{\partial \Sigma_2} \vec{R} d\vec{r}$$

če velja Stokesov izrek za Σ_1 in Σ_2

potem velja tudi za Σ

$$= \iint_{\partial \Sigma_1 - \Gamma} \vec{R} d\vec{r} + \int_{\Gamma} \vec{R} d\vec{r} + \int_{\Gamma} \vec{R} d\vec{r} + \int_{\partial \Sigma_2 - \Gamma} \vec{R} d\vec{r} = \int_{\partial \Sigma} \vec{R} d\vec{r}$$

$\underbrace{\quad}_{=0}$
 orientiran glede na Σ_1 orientiran glede na Σ_2

Greenova formula $\Rightarrow$ Stokesov izrek

Omejimo se lahko na primer, ko je

$$\Sigma \text{ graf nad } D \subseteq \mathbb{R}^2$$

$$\Sigma = \{(x, y, f(x, y)); x, y \in D \subseteq \mathbb{R}^2\}$$

$$f \in C^2(\bar{D})$$

Σ naj bo orientirana z normalo, ki kaže navzgor $\vec{N} = \frac{(-f_x, -f_y, 1)}{\sqrt{1+f_x^2+f_y^2}}$

$\partial \Sigma$ je graf f nad ∂D

Pozitivna orientacija ∂D nam da pozitivno (oz skladno) orientacijo $\partial \Sigma$

$$\vec{R} = (x, y, z)$$

$$\int_{\partial \Sigma} \vec{R} d\vec{r} = \int_{\partial \Sigma} x dx + y dy + z dz =$$

Na $\partial \Sigma$ je $z = f(x, y)$ torej

$$dz = f_x dx + f_y dy$$

$$= \int_{\partial D} x(x, y, f(x, y)) dx + y(x, y, f(x, y)) dy + z(f_x dx + f_y dy) =$$

$$= \int_{\partial D} (x + z f_x) dx + (y + z f_y) dy =$$

$$G.F. = \iint_D ((y + z f_y)_x - (x + z f_x)_y) dx dy =$$

$$= \iint_D (y_x + y_z f_x + z_x f_y + z_y f_x \cdot f_y + z f_{yx} - (x_y + x_z f_y + z_x f_x + z_z f_y \cdot f_x + z f_{xy})) dx dy$$

$$\text{rot } \vec{R} = (z_y - y_z, x_z - z_x, y_x - x_y)$$

$$= \iint_D (-f_x)(z_y - y_z) + (-f_y)(x_z - z_x) + (y_x - x_y) dx dy =$$

$$= \iint_D \frac{(-f_x, -f_y, 1)}{\sqrt{1+f_x^2+f_y^2}} \cdot (\text{rot } \vec{R})(x, y) \underbrace{\sqrt{1+f_x^2+f_y^2}}_{dS}$$

$$= \iint_{\Sigma} \text{rot } \vec{R} \cdot \vec{N} dS$$

Zgled :

$$\vec{R} = \frac{\vec{r}}{\|\vec{r}\|^3} \Rightarrow (-\frac{1}{r})$$

✓

19.3

primer

(ostalo je)

Gaussov rule ($n=3$)

Če gaussov vektor polja $\vec{v}$ v D_1 in $D_2 \Rightarrow$
velj za D

$$\iiint_D \operatorname{div} \vec{R} dV = \iiint_{D_1} \operatorname{div} \vec{R} dV + \iiint_{D_2} \operatorname{div} \vec{R} dV$$

Dokaz (Gaussova izreka)

ko je Ω teko oblike, da ved vsako od treh koordinatnih ravni leži med dvema grafoma $z = f(x, y)$ nad omejeno odprto množico s ploščo

$$\Omega: \vec{r} = (x, y, z)$$

$$\vec{N} = (N_x, N_y, N_z)$$

$$\iint_{\Omega} \vec{r} \cdot \vec{N} dS = \iiint_{\Omega} \text{div} \vec{r} dV$$

oziroma

$$\iint_{\partial\Omega} x(y_x + z_x + 2N_x) dS =$$

$$\stackrel{?}{=} \iiint_{\Omega} (x_x + y_y + z_z) dV$$

Oblike Ω implícite

$$\iint_{\partial\Omega} x N_x dS = \iiint_{\Omega} x_x dV$$

$$\iint_{\partial\Omega} y N_y dS = \iiint_{\Omega} y_y dV$$

$$\iint_{\partial\Omega} z N_z dS = \iiint_{\Omega} z_z dV$$



$$\iiint_{\Omega} z_z dV = \iint_D \left(\int_{g(x,y)}^{f(x,y)} z_z dz \right) dx dy =$$

$$= \iint_D (z(x, y, f(x, y)) - z(x, y, g(x, y))) dx dy$$

$$\iint_{\partial\Omega} z N_z dS = \iint_{\text{graf}(f)} z N_z dS + \iint_{\text{navpni}} z N_z dS + \iint_{\text{graf}(g)} z N_z dS$$

$$N = \frac{(-f_x, -f_y, 1)}{\sqrt{1+f_x^2+f_y^2}} \quad N_z = 0 \quad N_z = \frac{(g_x, g_y, -1)}{\sqrt{1+g_x^2+g_y^2}}$$

$$= \iint_D z(x, y, f(x, y)) \frac{1}{\sqrt{1+f_x^2+f_y^2}} \cdot \sqrt{1+f_x^2+f_y^2} dx dy +$$

$$+ 0 +$$

$$+ \iint_D z(x, y, g(x, y)) \frac{(-1)}{\sqrt{1+g_x^2+g_y^2}} \cdot \sqrt{1+g_x^2+g_y^2} dx dy$$

Greenovi identiteti

Naj bo $\Omega \subset \mathbb{R}^3$ kot v Gaussovem izreku

Naj bo $u, v \in C^2(\bar{\Omega})$

Tedaj velja:

$$1) \iint_{\partial\Omega} u \frac{dv}{d\vec{n}} dS = \iiint_{\Omega} (\vec{\nabla} u \cdot \vec{\nabla} v + \overset{\text{Laplace}}{u \Delta v}) dv$$

$$2) \iint_{\partial\Omega} \left(u \frac{dv}{d\vec{n}} - v \frac{du}{d\vec{n}} \right) dS = \iiint_{\Omega} (u \Delta v - v \Delta u) dv$$

Opombe: 1) Enostaka zunanja normala na $\partial\Omega$

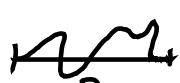
$$\begin{aligned} \therefore \frac{du}{d\vec{n}} &= \text{smerni odvod } u \text{ v smeri } \vec{n} \\ &= \vec{\nabla} u \cdot \vec{n} \end{aligned}$$

$$\dots) \frac{du}{d\vec{n}} dS = \vec{\nabla} u \cdot \vec{n} \cdot dS = \vec{\nabla} u \cdot d\vec{S}$$

....) "integracija po delih"

(....) u ; $\Delta u = 0$ harmonična

$$\Rightarrow \iiint_{\Omega} v \Delta u \, dv = 0$$

u ; $\forall v$  prikaže 0

$$\iiint_{\Omega} u \Delta v \, dv = 0 \quad \stackrel{?}{\Rightarrow} \quad u \text{ harmonična}$$

Dokaz:

$$\vec{R} = u \vec{\nabla} v = (u v_x, u v_y, u v_z) \neq$$

$$\begin{aligned} \operatorname{div} \vec{R} &= (u v_x)_x + (u v_y)_y + (u v_z)_z = \\ &= u_x v_x + u_y v_y + u_z v_z + u \Delta v = \\ &= \vec{\nabla} u \cdot \vec{\nabla} v + u \Delta v \end{aligned}$$

$$1) \quad \iint_{\partial \Omega} u \frac{dv}{dn} dS = \iint_{\partial \Omega} u \vec{\nabla} v \cdot \vec{n} \cdot dS = \iint_{\partial \Omega} \overbrace{u \vec{\nabla} v}^{\vec{R}} \cdot d\vec{S}$$

$$\begin{aligned} &\stackrel{\text{Gausov}}{=} \iiint_{\Omega} \operatorname{div}(u \vec{\nabla} v) dV = \iiint_{\Omega} (\vec{\nabla} u \cdot \vec{\nabla} v + u \Delta v) dV \end{aligned}$$

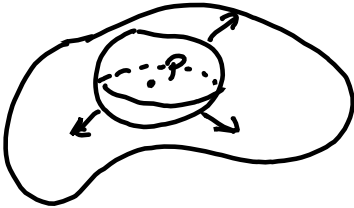
2) velja tudi:

$$\iint_{\partial \Omega} v \frac{du}{dn} dS = \iiint_{\Omega} (\vec{\nabla} v \cdot \vec{\nabla} u + v \Delta u) dV$$

o dostojenstvo

Brez koordinatne definicije divergence:

$$\vec{R}: D \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \quad C^1$$



$$P \in D; \quad \varepsilon > 0$$

$$\text{velja: } \iint_{S(P, \varepsilon)} \vec{R} d\vec{s} = \iiint_{K(P, \varepsilon)} \operatorname{div} \vec{R} dV$$

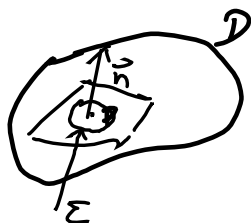
$$\frac{1}{V(K(P, \varepsilon))} \iint_{S(P, \varepsilon)} \vec{R} d\vec{s} = \frac{1}{V(K(P, \varepsilon))} \iiint_{K(P, \varepsilon)} \operatorname{div} \vec{R} dV$$

$\downarrow \varepsilon \rightarrow 0$

$$\lim_{\varepsilon \rightarrow 0} = \frac{3}{4\pi \varepsilon^3} \iint_{S(P, \varepsilon)} \vec{R} d\vec{s} = \operatorname{div} \vec{R}(P)$$

Pretok $\vec{R}$ skozi sfero

Brezkoordinatne definicije rotacije



$$p \in D$$

$$\vec{R}: D \rightarrow \mathbb{R}^3$$

$$|\vec{n}| = 1$$

Σ krog v ravnini; središčem p
in polmerom ϵ

$$\int_{\partial \Sigma} \vec{R} d\vec{r} = \iint_{\Sigma} \text{rot } \vec{R} \vec{n} dS$$

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\pi \epsilon^2} \int_{\partial \Sigma} \vec{R} d\vec{r} = (\text{rot } \vec{R})(p) \cdot \vec{n}$$

Diferencijalne forme $\Omega \subseteq \mathbb{R}^3$

0-forme: funkcije na Ω (vsaj C^2)

$u: \Omega \rightarrow \mathbb{R}$ odvod - diferencijal

$$du = u_x dx + u_y dy + u_z dz$$

$dx =$ diferencijal funkcije $(x, y, z) \mapsto x$

$$dy = -11-$$

$$dz = -11-$$

$$du(\vec{v}) = u_x v_1 + u_y v_2 + u_z v_3 = \vec{\nabla} u \cdot \vec{v}$$

$$\omega = X(x, y, z) dx + Y(x, y, z) dy + Z(x, y, z) dz$$

$$\omega(\vec{v}) = Xv_1 + Yv_2 + Zv_3$$

dx, dy, dz vsaki točki tvorijo bazu ~~funkcijske~~ funkcionalne

Egzaktne forme

ω je egzaktna $\Leftrightarrow$ je odvod 0-forme
 $\omega = du$

Forme mnożimo:

Produkt dwóch 1-form jest 2-forma

$\wedge$ vnanji produkt

•) Na 1-formah je antikomutativen

$$dx \wedge dy = -dy \wedge dx$$

$$dy \wedge dz = -dz \wedge dy$$

$$dz \wedge dx = -dx \wedge dz$$

$$\Rightarrow dx \wedge dx = -dx \wedge dx = 0 = dy \wedge dy = dz \wedge dz$$

••) Standardna baza prostora 2-form v

vseki točki je $(dy \wedge dz, dz \wedge dx, dx \wedge dy)$

Vseke 2-forma je oblike

$$V = A(x, y, z) dy \wedge dz + B(x, y, z) dz \wedge dx + C(x, y, z) dx \wedge dy$$

odwiod 1-forme

$$\omega = Xdx + Ydy + Zdz$$

$$d\omega = dX \wedge dx + dY \wedge dy + dZ \wedge dz =$$

$$= (X_x dx + X_y dy + X_z dz) \wedge dx + \\ + (Y_x dx + Y_y dy + Y_z dz) \wedge dy + \\ + \dots =$$

$$= -X_y dx \wedge dy + X_z dz \wedge dx + \\ + \cancel{Y_z dz \wedge dx} + Y_x dx \wedge dy \\ - Y_z dy \wedge dz - Z_x dx \wedge dx + Z_y dy \wedge dz =$$

$$d\omega = (Z_y - Y_z) dy \wedge dz + (X_z - Z_x) dz \wedge dx + \\ (Y_x - X_y) dx \wedge dy$$

1-forma ω je sklenjena če je $d\omega = 0$

• Vsaka eksaktna forma je sklenjena

$$d^2 = 0$$

$d_0: 0 \text{ forme} \rightarrow 1 \text{ forme}$

$d_1: 1 \text{ forme} \rightarrow 2 \text{ forme}$

$d_1 \circ d_0 = d^2$ oznaka

$$\int_{\vec{\Gamma}} du = \int_{\partial \vec{\Gamma}} u = \int_{\vec{\Gamma}} du$$

$$\int_{\partial \Sigma} \omega = \int_{\Sigma} d\omega$$

$$\int_{\partial \Sigma} \vec{K} d\vec{r} = \iint_{\Sigma} \text{rot} \vec{K} d\vec{S}$$

3. forma

$$V = u(x, y, z) dx \wedge dy \wedge dz =$$

$$= -u dy \wedge dx \wedge dz = u dy \wedge dz \wedge dx$$

2. forma

2. forma V je ekzaktna, če $\exists w$ 1. forme,
da je $V = dw$

2. forma V je sklenjena če je $dV = 0$

$$V = A dy \wedge dz + B dz \wedge dx + C dx \wedge dy$$

$$dV = dA \wedge dy \wedge dz + dB \wedge dz \wedge dx + dC \wedge dx \wedge dy$$

$$= (A_x dx + A_y dy + A_z dz) \wedge dy \wedge dz + \dots + \dots$$

$$= (A_x + B_y + C_z) dx \wedge dy \wedge dz$$

Vseke ekzaktna 2. forma je sklenjena
Odvod 3. forme je enak 0 (ker smo v 3D)

Vseke 3 forma je sklenjena

$$0 \xrightarrow{d_{-1}} 0\text{-forme} \xrightarrow{d_0} 1\text{-forme} \xrightarrow{d_1} 2\text{-forme} \xrightarrow{d_2} 3\text{-forme} \xrightarrow{d_3} 0$$

$\text{im } d_j \subseteq \ker d_{j+1}$

Sklenjene 0-forme

$$du = 0$$

ne vseki povezan; komponenti so je u konst.

Ω povezan: $\mathbb{R}$

$$\bigcirc \bigcirc : \mathbb{R}^2$$

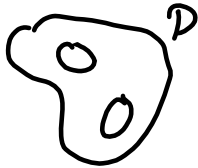
$\mathbb{R}^{\Omega(\Omega)}$ $\swarrow$ 57 komponent
za povezanost

Gaussov izrek:

$$\iint_{\partial \Omega} \vec{R} dS = \iiint_{\Omega} \operatorname{div} \vec{R} dV$$

$$\iint_{\partial \Omega} V = \iiint_{\Omega} dV$$

$\Omega \subseteq \mathbb{R}^2$
 $\nwarrow$ kompaktnost $D \subset \Omega$
 $D \subseteq \subseteq \Omega$



∂D sklenjena krivulja
 $\partial(\partial D) = \emptyset$

Γ krivulja
 $\partial \Gamma = \emptyset$ (je sklenjena krivulja)

$\exists D. \partial D = \Gamma$?

Y

24.3 huge um

$$\text{rot}\left(\frac{1}{H_j} y_j\right) = 0$$

$$\text{div}(\vec{r}) = \text{div}\left(R_1 H_2 H_3 \frac{1}{H_2} y_2 \times \frac{1}{H_3} \vec{y}_3 + \dots + \dots\right) =$$

$$= R_1 H_2 H_3 \text{div}\left(\frac{1}{H_2} y_2 \times \frac{1}{H_3} y_3\right) + \vec{\nabla}\left(R_1 H_2 H_3 \frac{1}{H_2} y_2 \times \frac{1}{H_3} y_3 + \dots + \dots\right)$$

$$= \vec{\nabla}\left(R_1 H_2 H_3\right) \frac{1}{H_2} y_2 \times \frac{1}{H_3} y_3 + \dots =$$

$$= \frac{1}{H_2 H_3} \nabla(R_1 H_2 H_3) \cdot y_1 + \frac{1}{H_1 H_3} \vec{\nabla}(H_1 R_2 H_3)$$

... ?

Triditer

$$\Delta u = \frac{1}{H_1 H_2 H_3} \left(\frac{\partial}{\partial u_1} \left(\frac{H_2 H_3}{H_1} \frac{\partial U}{\partial u_1} \right) + \right. \\ \left. + \frac{\partial}{\partial u_2} \left(\frac{H_1 H_3}{H_2} \frac{\partial U}{\partial u_2} \right) + \right. \\ \left. + \frac{\partial}{\partial u_3} \left(\frac{H_1 H_2}{H_3} \frac{\partial U}{\partial u_3} \right) \right)$$

Posledica:

1) Laplace v sferičnih koordinatah

$$(H_1 = 1, H_2 = r, H_3 = r \sin \varphi)$$

$$\Delta u = \frac{1}{r^2 \sin \varphi} \left(\frac{\partial}{\partial r} (r \sin \varphi \frac{\partial U}{\partial r}) + \right. \\ \left. + \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial U}{\partial \varphi} \right) + \right. \\ \left. + \frac{\partial}{\partial \varphi} \left(\frac{1}{\sin \varphi} \frac{\partial U}{\partial \varphi} \right) \right)$$

2) Laplace v cilindričnih koordinatah

$$(H_1 = 1, H_2 = r, H_3 = 1) \quad (r, \varphi, z)$$

$$\frac{1}{r} \left(\frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{r} \frac{\partial U}{\partial \varphi} \right) + \frac{\partial}{\partial z} \left(r \frac{\partial U}{\partial z} \right) \right)$$

3) Polarne koordinate v $\mathbb{R}^2$

$$\Delta U = \frac{1}{r} \left(\frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{r} \frac{\partial u}{\partial \varphi} \right) \right)$$

Zagled:

1) $\mathbb{R}^3$ $u(x, y, z)$ harmonične, radialno simetrične

$$u(x, y, z) = f(\sqrt{x^2 + y^2 + z^2}) = f(r)$$

$$\Delta u = 0 \Leftrightarrow \frac{d}{dr}(r^2 \sin u f'(r)) = 0$$

$$\Rightarrow (r^2 \cdot f'(r))' = 0$$

$$\Rightarrow r^2 \cdot f'(r) = A$$

$$f'(r) = \frac{A}{r^2}$$

$$f(r) = \frac{B}{r} + C$$

2) $\mathbb{R}^2$ $u(x, y)$ radialno simetrične harmonične
($f(r) = f(\sqrt{x^2 + y^2})$)

$$\Delta u, \frac{1}{r} \frac{d}{dr}(r f'(r)) = 0$$

$$\Rightarrow (r f'(r))' = 0$$

$$r(f'(r)) = A$$

$$f'(r) = \frac{A}{r}$$

$$f(r) = A \ln(r) + B$$

Kompleksna analiza

$(a, b) \in \mathbb{R}^2$
vredjamo množenje (poles seštanja po komp.)
 $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$

$(\mathbb{R}^2, +, \cdot)$ je komutativen obseg (parje)

$$(\mathbb{R}, +, \cdot) \leq (\mathbb{R}^2, +, \cdot)$$

$$a \mapsto (a, 0)$$

$$1 = (1, 0) \quad i^2 = (-1, 0)$$

$$i = (0, 1)$$

$$(a, b) = a + bi$$

Konjugiranje

$$z \mapsto \bar{z}$$

$$a + bi \mapsto a - bi$$

$(\mathbb{R}^2, i) = \mathbb{C} = G$ Gaussova ravnina (tudi kompleksna ravnina)

$$z = a + bi$$

$$a = \operatorname{Re} z = \frac{z + \bar{z}}{2}$$

$$b = \operatorname{Im} z = \frac{z - \bar{z}}{2i}$$

$$|z| = \sqrt{z \bar{z}} = \sqrt{a^2 + b^2}$$

Veljajo lastnosti

$$\cdot) |z| \geq 0 \quad \forall z$$

$$\cdot\cdot) |z| = 0 \Leftrightarrow z = 0$$

$$\cdot\cdot\cdot) |zw| = |z| |w| \quad \forall z, w$$

$$\cdot\cdot\cdot\cdot) \forall z, w. |z + w| \leq |z| + |w|$$

$$\cdot\cdot\cdot\cdot\cdot) |z - w| \geq ||z| - |w||$$

$$\cdot\cdot\cdot\cdot\cdot\cdot) |\bar{z}| = |z|$$

Metrike: $d(z, w) = |z - w|$

Deljenje

$$\frac{z}{w} = \frac{z \bar{w}}{w \cdot \bar{w}}$$

Polarni zapis kompleksnega števila:

$$z = x + yi \quad x, y \in \mathbb{R}$$

$$|z| = r \quad x = r \cos \varphi$$

$$y = r \sin \varphi$$

$$r \geq 0$$

$$\varphi \in [0, 2\pi)$$

$$\varphi = \arg(z)$$

$$z = r \cos \varphi + i r \sin \varphi = r e^{i\varphi} = |z| e^{i\varphi} = |z| e^{i \arg(z)}$$

Črnačke

•) D povezane odprta množica = območje

•) $\Delta(\alpha, r) = \{z \in \mathbb{C} ; |z - \alpha| < r\} = \text{disk}$
 $\alpha \in \mathbb{C}, r > 0$ središčem α , in polmerom r

$\Delta = \Delta(0, 1)$ enotski disk

•) $\Delta^*(\alpha, r) = \Delta(\alpha, r) - \{\alpha\} = \text{prebadena}$
(punktirana) okolica α

•) zaporedje: $z_n = \alpha_n + i\beta_n \quad n \in \mathbb{N}$
 $\alpha, \beta \in \mathbb{R}$

Zaporedje z_n konvergira $\Leftrightarrow$ konvergirata

α_n in β_n . limita je $\lim_{n \rightarrow \infty} \alpha_n + i \lim_{n \rightarrow \infty} \beta_n$

•) veznost: $f: E \subseteq \mathbb{C} \rightarrow \mathbb{C} \quad z_0 \in E$

f je vezna v z_0 , če za $\forall \varepsilon > 0, \exists \delta > 0$.

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$$

•) limite funkcij: $f: \Delta^*(\alpha, r) \rightarrow \mathbb{C}$
 f ima limito enako A ko gre $z \rightarrow \alpha$ ko
 $\forall \varepsilon > 0, \exists \delta > 0, (0 < \delta \leq r), \forall z \in \Delta^*(\alpha, \delta)$
 $\Rightarrow |f(z) - A| < \varepsilon$

• $f: D \subseteq \mathbb{C} \rightarrow \mathbb{C}$

$$z \mapsto f(z)$$

$\parallel$

$x + iy$

$\parallel$

$$u(z) + i v(z)$$

$$u(z), v(z) \in \mathbb{R}$$

$$u(x, y), v(x, y)$$

f lahko pišemo

$$f = u + iv$$

$$u, v: D \rightarrow \mathbb{R}$$

realni funkciji

Zagled

$$1) f(z) = z = x + iy$$

$$u(x, y) = x \quad v(x, y) = y$$

$$2) f(z) = z^2 = x^2 - y^2 - 2ixy$$

$$u(x, y) = x^2 - y^2 \quad v(x, y) = 2xy$$

$$3) f(z) = \bar{z} = x - iy$$

$$u(x, y) = x \quad v(x, y) = -y$$

$$f(z) = |z|^2 = z \cdot \bar{z} = x^2 + y^2$$

$$u(x, y) = x^2 + y^2 \quad v(x, y) = 0$$

$$\bullet) f = u + iv \quad f: E \rightarrow \mathbb{C}$$

$$f \text{ je zvezna v } z_0 \in E \iff \begin{matrix} u, v \text{ zvezni;} \\ v' z_0 \in E \end{matrix}$$

$$\bullet) f: D^*(\alpha, r) \rightarrow \mathbb{C}$$

$$f \text{ ima limito ko gre } z \text{ proti } \alpha \iff$$

$$\text{ima ta u in v limita ko gre } z \rightarrow \alpha$$

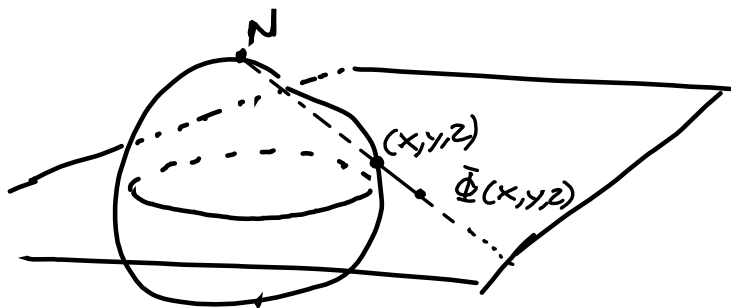
Torej

$$\lim_{z \rightarrow \alpha} f(z) = \lim_{z \rightarrow \alpha} u(z) + i \lim_{z \rightarrow \alpha} v(z)$$

U oči je ugodno si predstavljati da
je $\mathbb{C}$ del riemannove sfere

Kompaktifikacija $\mathbb{C}$ z eno točko

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$$



$$t \mapsto (0, 0, 1) + t(x, y, z - 1) = (tx, ty, 1 + t(z - 1))$$

$$\text{iscemo } 1 + t(z - 1) = 0 \Rightarrow t = \frac{1}{1 - z}$$

$$\Phi(x, y, z) = \frac{x}{1 - z} + \frac{yi}{1 - z} \quad \text{izmemorirajmo}$$

$$\Phi^{-1}(x + iy) = \frac{1}{1 + x^2 + y^2} (2x, 2y, x^2 + y^2 - 1)$$

$$\mathbb{C} \cup \{\infty\} \longrightarrow S^2$$

$$\Phi^{-1} \text{ ne } \Phi$$

$$\Phi^{-1}(\infty) = N$$

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \longleftarrow S^2$$

$\hookrightarrow$ take topologija da je to homeomorfizem

ke j so delice tocke ∞ ? (Baze delice)

$$\mathbb{C} = \overline{\Delta(0, R)} \quad \text{z } 0 \text{ je se levi:}$$

Homeomorfne funkcije

Naj bo $D \subseteq \mathbb{C}$ $\alpha \in D$. $F: D \rightarrow \mathbb{C}$

Funkcija F je v kompleksnem smislu

odvedljiva v α , če obstaja limita

$$\lim_{z \rightarrow \alpha} \frac{f(z) - f(\alpha)}{z - \alpha} = \lim_{h \rightarrow 0} \frac{f(\alpha + h) - f(\alpha)}{h} = f'(\alpha)$$

če ta limita obstaja je $f'(\alpha)$ kompleksni:
odvod f v α

Zgled

1) $f(z) = z$

$$\lim_{h \rightarrow 0} \frac{f(\alpha + h) - f(\alpha)}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1 = f'(\alpha)$$

2) $f(z) = z^2$

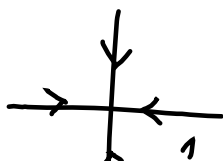
$$\lim_{h \rightarrow 0} \frac{(\alpha + h)^2 - \alpha^2}{h} = \lim_{h \rightarrow 0} \frac{2\alpha h + h^2}{h} = 2\alpha$$

3) $f(z) = \bar{z}$

$$\lim_{h \rightarrow 0} \frac{(\alpha + h) - \alpha}{h} = \lim_{h \rightarrow 0} \frac{\bar{h}}{h}$$

$$h \in \mathbb{R} - \{0\} \Rightarrow \frac{\bar{h}}{h} = 1$$

$$h \in i\mathbb{R} - \{0\} \Rightarrow \frac{\bar{h}}{h} = -1$$



Torej ta limita ne obstaja

$$\mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$(x, y) \mapsto (x, -y)$ linearna preslikava

je $C^\infty(\mathbb{R}^2)$, ampak v kompleksnem
smislu ni odvedljiva

Ni v kompleksnem smislu odvedljiva
v nobeni točki

4) $f(z) = |z|^2 = z \cdot \bar{z}$

$$\alpha \in \mathbb{C}$$

$$\lim_{h \rightarrow 0} \frac{f(\alpha + h) - f(\alpha)}{h} = \lim_{h \rightarrow 0} \frac{\bar{\alpha}h + \alpha\bar{h} + h\bar{h}}{h} =$$

$$= \lim_{h \rightarrow 0} \left(\bar{\alpha} + \alpha \frac{\bar{h}}{h} + \bar{h} \right)$$

Ta limita obstaja $\Leftrightarrow \alpha = 0$

Opomba:

1) Odvedljivost nad $\mathbb{R}$ ne pomeni nujno odvedljivosti v $\mathbb{C}$

2) Izkazuje se: $\mathbb{C}$ je f v vsaki točki

$\alpha \in D$ v kompleksnem smislu odvedljiv

$\Rightarrow f$ je neskončnokrat odvedljiv

$$f \in C^\infty(D)$$

Naj bo f odvedljiva v kompleksnem smislu; $\alpha \in D$ ($f: D \rightarrow \mathbb{C}$)

$$\lim_{z \rightarrow \alpha} \frac{f(z) - f(\alpha)}{z - \alpha} = f'(\alpha)$$

$$\frac{f(z) - f(\alpha)}{z - \alpha} - f'(\alpha) = \gamma(z - \alpha)$$

$$\frac{f(\alpha + h) - f(\alpha)}{h} - f'(\alpha) = \gamma(h)$$

$$\lim_{h \rightarrow 0} \gamma(h) = 0$$

$$f(\alpha + h) = f(\alpha) + f'(\alpha) \cdot h + h\gamma(h)$$

Toditev:

Naj bo $f: D \subset \mathbb{C} \rightarrow \mathbb{C}$ v kompleksnem smislu odvodljiva v $\alpha \in D$. Torej:

1) f zmeva α

2) f je diferenciable v α in

$$Df(\alpha)h = f'(\alpha) \cdot h$$

$$h \mapsto w \cdot h \quad w \in \mathbb{C} \quad w = a + ib \quad h = h_1 + h_2 i$$

$$w \cdot h = (ah_1 - bh_2) + i(ah_2 + bh_1)$$

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \mapsto \begin{bmatrix} ah_1 - bh_2 \\ bh_1 + ah_2 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

Def:

Če je f v vsaki točki $z \in D$ v kompleksnem smislu odvedljiva je f holomorfná nad D

✓ 7.4

f def

$$f \in O(D) \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0 \text{ in } D$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) = \frac{1}{2} (u_x + i u_x) - i(u_y + i u_y) =$$

$$\frac{1}{2} (u_x + i(-u_y) - u_y + 1 \cdot u_x) = u_x - i u_y$$

$$f \in O(D) \Rightarrow \frac{\partial f}{\partial \bar{z}} = 0$$

$$\frac{\partial f}{\partial \bar{z}} = f'$$

zaged

$$f(z) = z$$

$$\frac{\partial f}{\partial z} = 1$$

$$f(z) = z^2$$

$$\frac{\partial f}{\partial z} = 2z$$

$$f(z) = z \bar{z}$$

$$\frac{\partial f}{\partial z} = \bar{z}$$

$$\frac{\partial f}{\partial \bar{z}} = z$$

$$A: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad \mathbb{R}\text{-linearne} \quad *$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\mathbb{C} \longrightarrow \mathbb{C} \quad \text{kde } j \text{ je } A \text{ } \mathbb{C}\text{-linearne}$$

$$A(iv) = iA(v) \quad ? \quad **$$

$$* + ** = \mathbb{C}\text{-linearne}$$

$$z \mapsto iz \text{ je lineárna preskita}$$

$$x + iy \mapsto -y + ix$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ x \end{bmatrix} \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$i^2 = -1 \quad J^2 = -I$$

$$A \text{ je } \mathbb{C}\text{-linearne} \iff AJ = JA$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} b & -a \\ d & -c \end{bmatrix} = \begin{bmatrix} c & -d \\ a & b \end{bmatrix}$$

$$b = -c \\ d = a$$

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

Trditev: Naj bo f diferenciable na $D \subseteq \mathbb{C}$
 Torej je f holomorfna nad D ($f \in O(D)$)
 $\Leftrightarrow df$ je v vsaki točki $\mathbb{C}$ -linearen

Dokaz:

$$f = u + iv$$

$$df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

če $f \in O(D) \Leftrightarrow$ velja CR sistem

$$u_x = v_y$$

$$u_y = -v_x$$

$$\Leftrightarrow \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \text{ } \mathbb{C} \text{ linearne}$$

$$\parallel$$

$$\begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix}$$

$$f' = u_x - i u_y$$

$$z \mapsto f'(\alpha) \cdot z \quad [f'(\alpha)]$$

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \quad \left. \vphantom{\frac{\partial f}{\partial z}} \right\} +$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \quad \left. \vphantom{\frac{\partial f}{\partial \bar{z}}} \right\} -$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} \quad -; \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}$$

$$\frac{\partial f}{\partial y} = i \left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}} \right)$$

$$df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

$$df \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x u_x + y u_y \\ x v_x + y v_y \end{bmatrix} =$$

$$= \begin{bmatrix} u_x \\ v_x \end{bmatrix} x + \begin{bmatrix} u_y \\ v_y \end{bmatrix} y = f_x x + f_y y =$$

$$z = x + yi$$

$$\bar{z} = x - yi$$

$$x = \frac{1}{2}(z + \bar{z}) \quad y = \frac{1}{2i}(z - \bar{z})$$

$$= (f_z + f_{\bar{z}}) \cdot \frac{1}{2}(z + \bar{z}) + i (f_z - f_{\bar{z}}) \frac{1}{2i}(z - \bar{z}) =$$

$$= f_z \cdot z + f_{\bar{z}} \bar{z}$$

$$df(z) = f_z(x)z + f_{\bar{z}}(x)\bar{z}$$

$\nwarrow$ linear $\nwarrow$ antilinear

Vsebo r-linearno preslikavo lahko enolično
 razpišemo kot vsoto linearne in
 antilinearne preslikave

$\bar{c}$ je f diferenciable na D je a

$$\forall \alpha \in D$$

$$f(\alpha+h) = f(\alpha) + df_{\alpha} h + o(h)$$

$$f(\alpha+h) = f(\alpha) + f_z(\alpha)h + f_{\bar{z}}(\alpha)\bar{h} + o(h)$$
$$\lim_{h \rightarrow 0} \frac{|o(h)|}{|h|} = 0$$

$$\lim_{h \rightarrow 0} \frac{f(\alpha+h) - f(\alpha)}{h} = \lim_{h \rightarrow 0} f_z(\alpha) + f_{\bar{z}}(\alpha) \frac{\bar{h}}{h} + \frac{o(h)}{h}$$

$$\text{obstoj} \Leftrightarrow f_{\bar{z}}(\alpha) = 0$$

Potence vrste v kompleksnem $\mathbb{C}$

$$a_0 + a_1 z + \dots + a_n z^n \in O(\mathbb{C})$$

Številске vrste:

$$S = a_1 + \dots = \lim S_n$$

$$S_n = a_1 + \dots + a_n$$

Vrste konvergira absolutno če

$$\text{konvergira } \sum_{n=1}^{\infty} |a_n|$$

Torej vrste konvergira

Funkcijske vrste in zaporedja

$$f_n: D \subset \mathbb{C} \rightarrow \mathbb{R}$$

$$F = f_1 + \dots + f_n + \dots$$

- konvergenca po točki:

$$\forall z \in D. f_1(z) + f_2(z) + \dots$$

- enakomerna konvergenca na D

$$\text{vrsta } \sum_{i=0}^{\infty} a_n f^n \text{ konvergiira enakomerno}$$

$$\text{če za } \forall \epsilon > 0. \exists N > 0. \forall n > N_0$$

$$\left| \sum_{j=0}^n f_j(z) - F(z) \right| < \epsilon \quad \forall z \in D$$

- enakomerna konvergenca na kompaktnih množicah

$$\text{vrsta } \sum_{j=0}^{\infty} f_j \text{ konvergiira enakomerno na kompaktnih}$$

$$\text{množicah } D \text{ proti } F, \text{ če } \forall K \subset D^{\text{komp}}$$

$$\forall \epsilon > 0. \exists N_0 \in \mathbb{N} \forall n > N_0 \forall k \in K.$$

$$\left| \sum_{j=0}^n f_j(z) - F(z) \right| < \epsilon \quad \forall z \in K$$

$$f_1(z) = 1$$

$$f_2(z) = z - 1$$

$$f_3(z) = z^2 - z$$

$\vdots$

$$f_n(z) = z^{n-1} - z^{n-2}$$

$$S_n(z) = z^{n-1}$$

konvergiira enakomerno na kompaktnih podmnožicah enotnega diska k funkciji 0

$$K^{\text{komp}} \subset \Delta$$

$$\exists 0 < \rho < 1$$

$$K \subset \Delta(0, \rho)$$

$$|S_N(z) - 0| = |z^{N-1}| \leq \rho^{N-1}$$

Tud: ~~imply~~ Weierstrass criterion

$$f_j: K \longrightarrow \mathbb{C}$$

$$\forall j \exists M_j. (f_j(z) \leq M_j) \text{ za } \forall z \in K$$

$$\text{in } \sum_1^{\infty} M_j < \infty$$

$$\Rightarrow \sum_1^{\infty} f_j \text{ konvergiere enakomerne na } K$$

Potence vrste

$$a_0, a_1, \dots \in \mathbb{C}$$

$$\sum a_j (z - \alpha)^j \quad \text{potence vrste s središčem}$$

v α in koeficienti $a_0, a_1, \dots$

Iskaj

obstaj tak $R \in [0, \infty]$ da potence

vrsta $\sum a_j (z - \alpha)^j$ divergira za $\forall$

$z: |z - \alpha| > R$ in absolutno konvergira

za $\forall z: |z - \alpha| < R$ ($\Delta(\alpha, R)$)

za $\forall \alpha, r < R$ vrsta konvergira

enakomerno na $\overline{\Delta(\alpha, r)}$

Oz. ta vrsta konvergira enakomerno

na kompaktnih podmnožicah $\Delta(\alpha, R)$

Pri tem je $\frac{1}{R} = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$

Opomba: 1) $R = 0 \Leftrightarrow \limsup \sqrt[n]{|a_n|} = \infty$

2) $R = \infty \Leftrightarrow \limsup \sqrt[n]{|a_n|} = 0$

(1)

Zgled:

$$1) \sum_1^n \frac{z^n}{n^n} \quad \sqrt[n]{|a_n|} = \frac{1}{n}$$
$$\frac{1}{R} = 0 = \lim \frac{1}{n}$$
$$R = \infty$$

$$\sum_0^\infty \frac{z^n}{n!} \quad \lim \sqrt[n]{\frac{1}{n!}} = 0$$

$$2) \sum_1^\infty n^n z^n \quad R=0$$

$$3) \sum_0^\infty z^n \quad R=1$$

$|z|=1$ divergira

$$4) \sum \frac{z^2}{n^2} \quad R=1$$

$\forall |z|=1$ konvergira

$$5) \sum \frac{z^n}{n} \quad R=1$$

$z=1 \Rightarrow |z|=1$ divergira

$z=-1 \Rightarrow$ konvergira

Opomba:

$$\sum_1^N \frac{z^n}{n} = \frac{z}{1-z} \left(\sum_1^{N-1} \frac{1-z^n}{n(n+1)} + \frac{1-z^N}{N} \right)$$

$|z| \neq 1$

ko $|z|=1$

to je konvergira po vsaki strani u $z=1$

$$\limsup_{n \rightarrow \infty} b_n = b \quad b < \infty$$

$$\forall \varepsilon > 0. \quad \begin{array}{c} b-\varepsilon \quad b \quad b+\varepsilon \\ | \quad | \quad | \\ \hline \end{array}$$

členov zaporedja b_n , ki so večji od $b+\varepsilon$ je končno mnogo in členov zaporedja, ki so večji od $b-\varepsilon$ je neskončno mnogo

Dokaz (trditve)

$$\text{BSZS: } \alpha = 0 \quad \sum a_n z^n$$

$$\text{Naj bo } |z| > R \Leftrightarrow \frac{1}{R} > \frac{1}{|z|}$$

$$\limsup \sqrt[n]{a_n} > \frac{1}{|z|}$$

Obstaja neskončno mnogo indeksov n , da je $\sqrt[n]{|a_n|} > \frac{1}{|z|}$

$$|a_n z^n| > 1 \Rightarrow \text{vrsta divergira}$$

$$|z| < R$$

$$|z| < \rho < R \quad \overline{\Delta(0, r)}$$

$$|z| < r < R$$

$$\limsup \sqrt[n]{a_n} = \frac{1}{R} < \frac{1}{r} < \frac{1}{|z|}$$

$$\exists n_0. \forall n > n_0. \sqrt[n]{a_n} < \frac{1}{r} < \frac{1}{|z|}$$

$$\alpha \quad |a_n z^n| < \frac{|z|^n}{r^n} < \left(\frac{\rho}{r}\right)^n < 1$$

$$\sum_{n=0}^{\infty} a_n z^n \leq \sum_{n=0}^{\infty} \left(\frac{\rho}{r}\right)^n < \infty$$

vrsta konvergira enakomerno na

$$\overline{\Delta(0, \rho)}$$

in absolutno

Definicija:

Naj bo $f: D \subset \mathbb{C} \rightarrow \mathbb{C}$

Naj bo $\alpha \in D$. f se da v okolici α
razviti v potenčno vrsto če $\exists r > 0$

in potenčno vrsto $\sum_{n=0}^{\infty} a_n (z-\alpha)^n$ da ne

$\triangle (\alpha, r)$ velja $\forall z$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n$$

(konvergenčni radij je vsaj r)

Trditav: Naj bo

$$f(z) = \sum_0^{\infty} a_n (z-\alpha)^n \text{ na } \Delta(\alpha, r) \text{ } r > 0.$$

Torej je $f \in O(\Delta(\alpha, r))$ in velja

$$f'(z) = \sum_1^{\infty} a_n \cdot n (z-\alpha)^{n-1}$$

Posledica: če sedaj $f: D \rightarrow \mathbb{C}$ v okolici
vsake točke $\alpha \in D$ razviti v potenčno
vrsto je $f \in O(D)$

Posledica: če je $f(z) = \sum a_n (z-\alpha)^n$ na

$\Delta(\alpha, r)$, $r > 0$, so tudi vsi njeni odvodi
holomorfnе funkcije na $\Delta(\alpha, r)$
pri tem je $f'(z) = \sum_1^{\infty} n a_n (z-\alpha)^{n-1}$

$$f''(z) = \sum_2^{\infty} n(n-1) a_n (z-\alpha)^{n-2}$$

$$f(\alpha) = a_0$$

$$f'(\alpha) = a_1$$

$$f''(\alpha) = 2a_2$$

$$f^{(3)}(\alpha) = 6a_3$$

$$f^{(n)}(\alpha) = n! a_n$$

$$a_n = \frac{f^{(n)}(\alpha)}{n!}$$

$$f(z) = \sum_0^{\infty} \frac{f^{(n)}(\alpha)}{n!} (z-\alpha)^n$$

12.4

Dokaz:

$$0 < r < R \quad \text{vrsta} \quad \sum_0^{\infty} a_n (z-\alpha)^n$$

ds ne vidim kaj pise ne
tudi in abgujem