

Fourierova vrsta

$$FV(f)(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$\uparrow$
 $\frac{a_0}{2}$ na predavanjih

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

$FV(f)$ konvergira k f , če je f zvezna v x ,
če pa ni, pa konvergira k $\frac{f(x^-) + f(x^+)}{2}$

$$\int x \cos(nx) dx = \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} + C$$

$$\int x \sin(nx) dx = -\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} + C$$

$$f \text{ sode} \Rightarrow b_n = 0$$

$$f \text{ liha} \Rightarrow a_n = 0$$

$$\int x^2 \cos(nx) dx = \frac{x^2}{n} \sin(nx) + \frac{2x}{n^2} \cos(nx) - \frac{2}{n^3} \sin(nx)$$

$$\int x^2 \sin(nx) dx = -\frac{x^2}{n} \cos(nx) + \frac{2x}{n^2} \sin(nx) + \frac{2}{n^3} \cos(nx)$$

②

$f(x) = |x|$ razvij v Fourierovo vrsto in
seštej $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx = \frac{1}{\pi} \int_0^{\pi} x dx = \frac{1}{2\pi} \pi^2 = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx =$$

sodast

$$= \frac{2}{\pi} \left(\frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right) \Big|_0^{\pi} =$$



$$= \frac{2}{\pi} \left(\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin(nx) dx = 0$$

lihost

$$FV(f)(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi} \left(\frac{(-1)^n - 1}{n^2} \right) \cos(nx) =$$

$$= \frac{\pi}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{-2}{(2k+1)^2} \cos((2k+1)x)$$

$$FV(f)(0) = \frac{\pi}{2} + \frac{-4}{\pi} \cdot \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = f(0) = 0$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = -\frac{\pi}{2} \cdot \frac{\pi}{(-4)} = \frac{\pi^2}{8}$$

Dodajmo

$$\Sigma = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = ? \quad S + \frac{1}{2^2} + \frac{1}{4^2} + \dots =$$

$S \dots$ lič.

$S' \dots$ ostalo

$$= S + \frac{1}{2^2} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right) =$$

$$= S + \frac{1}{4} \Sigma = \Sigma$$

$$S = \frac{3}{4} \Sigma$$

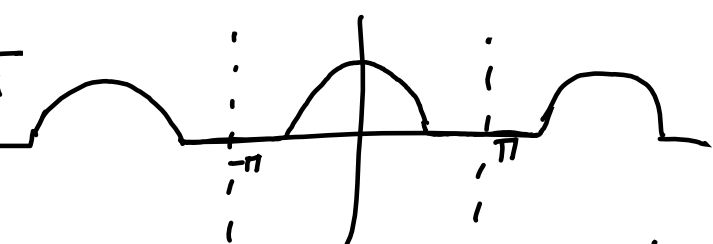
$$\Sigma = \frac{4}{3} S = \frac{\pi^2}{6}$$

③

$$f(x) = \max(\cos x, 0)$$

$$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}$$

$$S_2 = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$$



$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi/2} \cos(x) dx = \frac{1}{\pi} \sin\left(\frac{\pi}{2}\right) = \frac{1}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos(nx) dx = \frac{1}{\pi} \int_0^{\pi/2} \cos(x) \cos(nx) dx$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \cos(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi/2} \cos x \cos nx dx = \frac{2}{\pi} \int_0^{\pi/2} \cos x \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \frac{1}{2} (\cos((n+1)x) + \cos((n-1)x)) dx$$

$$\frac{1}{\pi} \left[\frac{1}{n+1} \sin\left((n+1)\frac{\pi}{2}\right) + \frac{1}{n-1} \sin\left((n-1)\frac{\pi}{2}\right) \right]$$

$$n=4k: \frac{1}{\pi} \frac{1}{n+1} \cdot 1 + \frac{1}{n-1} \cdot (-1) = \frac{-2}{n^2-1} \cdot \frac{1}{\pi} = \frac{-2}{16k^2-1} \cdot \frac{1}{\pi}$$

$$n \neq 4k: 0$$

$$n=4k+2: \frac{1}{\pi} \frac{1}{4k+3} \sin \frac{3\pi}{2} + \frac{1}{\pi} \frac{1}{4k+1} \sin \frac{\pi}{2} =$$

$$= \frac{1}{\pi} \left(-\frac{1}{4k+3} + \frac{1}{4k+1} \right)$$

$$= \frac{1}{\pi} \begin{cases} 0 & ; n \neq 4k \\ \frac{(-1)^{m+1} \cdot 2}{(2n)^2 - 1} & ; n = 2m \end{cases}$$

$$a_1 = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos^2 x dx = \frac{2}{\pi} \int_0^{\pi/2} \frac{1+\cos 2x}{2} dx = \frac{1}{\pi} + \frac{\sin(2x)}{2} \Big|_0^{\pi/2} = \frac{1}{2}$$

$$b_n = 0 \text{ ker } f \text{ soda}$$

$$FV(f)(x) = \frac{1}{\pi} + \frac{1}{2} \cos(x) + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \cdot 2}{(2m)^2 - 1} \cos(2mx)$$

$$1 = f(0) = FV(f)(0) = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi} S_1 \Rightarrow$$

$$S_1 = \left(\frac{1}{2} - \frac{1}{\pi} \right) \cdot \frac{\pi}{2} (-1) = \frac{1}{2} - \frac{\pi}{4}$$

$$f\left(\frac{\pi}{2}\right) = 0 = \frac{1}{\pi} + 0 - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)^2 - 1} \cdot \cos(2m\pi) =$$

$$= \frac{1}{\pi} - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{4m^2 - 1}$$

$$S_2 = -\frac{1}{\pi} \cdot \frac{\pi}{2} = -\frac{1}{2}$$

⑥

$$f(x) = x^2 \quad f: [0, \pi] \rightarrow \mathbb{R}$$

a) Razvij v kosinusno FV in
skiciraj graf

b) razvij v sinusno FV in skiciraj
njegov graf

1) če f razširimo do sode funkcije

$$f_s: [-\pi, \pi] \rightarrow \mathbb{R}$$

$$x < 0 \Rightarrow f_s(x) = f(-x)$$

$$FV_{\cos}(f)(x) = FV(f)(x)$$

$$2) f_l: [-\pi, \pi] \rightarrow \mathbb{R} \quad f_l(x) = -f(-x)$$

$$FV_{\sin}(f)(x) = FV(f_l)(x)$$

$$1) f_s(x) = x^2$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi} \int_0^{\pi} x^2 dx = \frac{\pi^2}{3}$$

$$b_n = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) dx$$

Preglejmo:

$$\int x^2 \cdot e^{inx} =$$

$$u = x^2 \quad dv = e^{inx} dx$$

$$du = 2x dx \quad v = \frac{1}{in} e^{inx}$$

$$= x^2 \cdot \frac{e^{inx}}{in} - \int 2x \frac{e^{inx}}{in} dx = \text{od prej}$$

$$= x^2 \cdot \frac{e^{inx}}{in} - \frac{2}{in} \left(\frac{ix e^{inx}}{n} + \frac{e^{inx}}{n^2} \right) + C$$

$$= e^{inx} \left(\frac{ix^2}{-n} + \frac{2x}{n^2} + \frac{2i}{n^3} \right)$$

$\cos nx + i \sin nx$

$$\int x^2 \cos(nx) dx = \frac{x^2}{n} \sin(nx) + \frac{2x}{n^2} \cos(nx) - \frac{2}{n^3} \sin(nx)$$

$$\int x^2 \sin(nx) dx = -\frac{x^2}{n} \cos nx + \frac{2x}{n^2} \sin(nx) + \frac{2}{n^3} \cos(nx)$$

$$a_n = \frac{1}{\pi} \left(\frac{x^2}{n} \sin(nx) + \frac{2x}{n^2} \cos(nx) + \frac{2}{n^3} \sin(nx) \right) \Big|_{-\pi}^{\pi} =$$

$$= \frac{2}{\pi} \left(\frac{2\pi}{n^2} (-1)^n \right) = \frac{4(-1)^n}{n^2}$$

$$FV_{\cos}(f)(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 4}{n^2} \cos(x)$$

$$b) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_{\theta}(x) \sin(nx) \, dx = \frac{2}{\pi} \underbrace{\int_0^{\pi} f_{\theta}(x) \sin(nx) \, dx}_{= f(x) \text{ on } [0, \pi] = x^2}$$

$$= \frac{2}{\pi} \left[-\frac{x^2}{n} \cos(nx) + \frac{2}{n^2} \sin(nx) + \frac{2}{3n} (\cos(nx)) \right] \Big|_0^{\pi}$$

$$= \frac{2}{\pi} \left(-\frac{\pi^2}{n} (-1)^n + \frac{2}{n^2} (-1)^n - \frac{2}{n^3} \right)$$

$$FV_{\sin x}(f)(x) = \sum_{n=1}^{\infty} \frac{2}{\pi} \left((-1)^{n+1} \frac{\pi^2}{2} + \frac{2(1 - (-1)^n)}{n^3} \right) \sin(nx)$$

$$f(x) = x(\pi - x) \quad \text{reuzij}$$

6)

$$f(x) = \sin^3 x \quad \text{razvij v FV}$$

Pred premislek:

$$f(x) = \sin^2 x \quad \text{je že FV}$$

$$b_2 = 1, \quad a_2 = 0$$

$$f(x) = \sin^2 x = \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta)) =$$

$$= \frac{1}{2} (\cos(0) - \cos(2x)) = \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$f(x) = \sin^2 x \cdot \sin x = \frac{1}{2} \sin x - \frac{1}{2} \cos 2x \sin x =$$

$$= \frac{1}{2} \sin x - \frac{1}{2} \cdot \frac{1}{2} (\sin(3x) - \sin(x)) =$$

$$= \frac{3}{4} \sin x - \frac{1}{4} \sin(3x)$$

polinom
 $\sqrt{p}(\sin x, \cos x)$ ima končno furierovo vrsto
 $p \in \mathbb{R}[x, y]$

$a > 0$ kvadrantnej težišče homogénega

loke astroide



$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

$$x_T = \frac{\int_K x dm}{m(K)} = \frac{\int_K x \rho ds}{\int_K \rho ds} = l(K)$$

$\vec{r}(t) \dots$ parametrizacija

$$ds = |\dot{\vec{r}}(t)| dt = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt$$

$u, \dots$ skalarná funkcia

$$\int_K u ds = \int_a^b u(\vec{r}(t)) |\dot{\vec{r}}(t)| dt$$

$$x = a \cos^3 t$$

$$y = a \sin^3 t$$

$t \in [0, \frac{\pi}{2}] \leftarrow$ keď je vše pozitívno (vyššie pravidlo)

$$\vec{r}(t) = (-3a \cos^2 t \sin t, 3a \sin^2 t \cos t)$$

$$|\dot{\vec{r}}| = 3a \sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t} = 3a \cos t \sin t$$

$$l(K) = \int_0^{\frac{\pi}{2}} |\dot{\vec{r}}(t)| dt = \int_0^{\frac{\pi}{2}} 3a \cos t \sin t dt =$$

$$u = \sin t dt$$

$$du = \cos t dt$$

$$= 3a \int_0^1 u du = \frac{3}{2} a$$



$$\int_K x ds = \int_0^{\frac{\pi}{2}} a \cos^3 t \cdot 3a \cos t \sin t dt =$$

$$\cos t = u \quad du = -\sin t dt$$

$$= 3a^2 \int_0^1 u^4 du = \frac{3}{5} a^2$$

$$y_T = \frac{3}{5} a^2$$

$$a > 0 \quad \alpha \in [0, 2\pi]$$

$$K = S(0, a) \cap \Pi$$

$$\Pi: y = x \tan \alpha$$

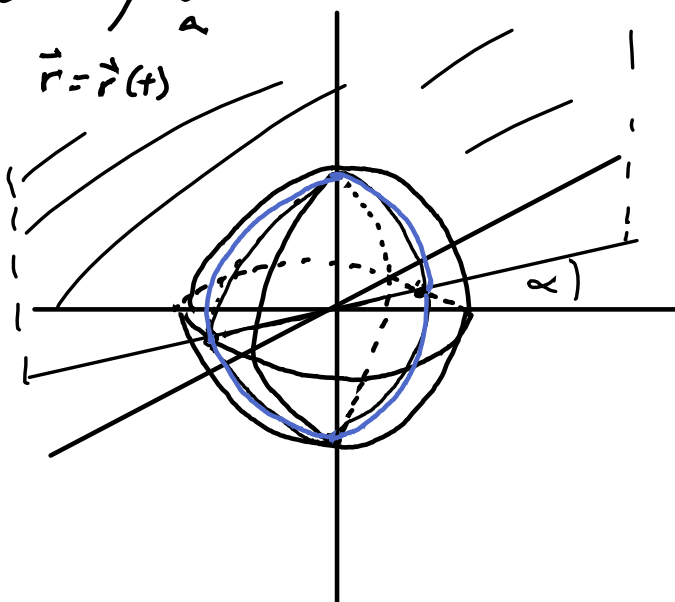
$$I = \int_K (y-z) + (z-x) dy + (x-y) dz$$

$$\vec{f}(x, y, z) = (y-z, z-x, x-y) \quad \text{vektor sho po'ye}$$

$$I = \int_K \vec{f} d\vec{r}$$

$$\vec{r} = (x, y, z) \Rightarrow d\vec{r}$$

$$\int_a^b \vec{f} d\vec{r} = \int_a^b \vec{f}(\vec{r}(t)) \dot{\vec{r}}(t) dt$$



Parametrizatsiya

$$x = a \cos \alpha \cdot \cos \vartheta$$

$$y = a \sin \alpha \cdot \cos \vartheta$$

$$z = a \sin \vartheta$$

$$\vec{r}(\vartheta) = (a \cos \vartheta \cos \alpha, a \cos \vartheta \sin \alpha, a \sin \vartheta)$$

$$\vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$\dot{\vec{r}}(\vartheta) = (-a \sin \vartheta \cos \alpha, -a \sin \vartheta \sin \alpha, a \cos \vartheta)$$

$$\vec{f}(\vec{r}(\vartheta)) = a (\cos \vartheta \sin \alpha - \sin \vartheta, \sin \vartheta - \cos \vartheta \cos \alpha, \cos \vartheta \sin \alpha - \cos \vartheta \cos \alpha)$$

$$\vec{f} \cdot \dot{\vec{r}} = a^2 (-\cos \alpha \cos \vartheta \sin \alpha \sin \vartheta + \sin^2 \vartheta \cos \alpha \cos \vartheta -$$

$$- \sin^2 \vartheta \sin \alpha + \cos \vartheta \cos \alpha \sin \vartheta \sin \alpha +$$

$$+ \cos^2 \vartheta \sin \alpha - \cos^2 \vartheta \cos \alpha) =$$

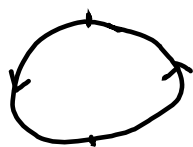
$$= a^2 (\cos 2\vartheta \sin \alpha - \cos 2\vartheta \cos \alpha) \quad \text{Me}$$

$$= a^2 (\cos \alpha - \sin \alpha)$$

$$\int_K \vec{f} d\vec{r} = \int_0^{2\pi} a^2 (\cos \alpha - \sin \alpha) d\vartheta =$$

$$2\pi a^2 (\cos \alpha - \sin \alpha)$$

↑ pri. orientatsiya



$$a, b, c \in \mathbb{R}$$

$$I = \int_S a \, dy \, dz + b \, dx \, dz + c \, dx \, dy$$

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 1, x, y, z \geq 0\}$$

$$I = \int \vec{f} \, d\vec{S}$$

$$\vec{f} = (a, b, c)$$

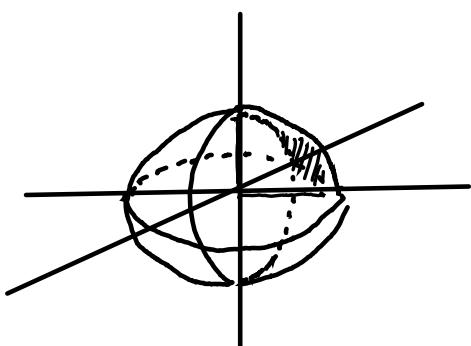
$$\int (\vec{f} \cdot \vec{N}) \, dS$$

$\nwarrow$ *enotska normala*
orientacija ploške je usmerjena s smerjom normale

$$\int_S \vec{f} \, d\vec{S} = \int_0 \vec{f}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \, du \, dv$$

$\nearrow$ $\vec{r} = \vec{r}(u, v)$

$\nwarrow$ smer se ujema s predpisano orientacijo



Sferične koordinate

$$x = \cos \vartheta \cos \varphi \quad \vartheta \in [0, \frac{\pi}{2}]$$

$$y = \cos \vartheta \sin \varphi \quad \varphi \in [0, \frac{\pi}{2}]$$

$$z = \sin \vartheta$$

$$\vec{r}(\vartheta, \varphi) = (\cos \vartheta \cos \varphi, \cos \vartheta \sin \varphi, \sin \vartheta)$$

$$\vec{r}_\vartheta = (-\sin \vartheta \cos \varphi, -\sin \vartheta \sin \varphi, \cos \vartheta)$$

$$\vec{r}_\varphi = (-\cos \vartheta \sin \varphi, \cos \vartheta \cos \varphi, 0)$$

$$\vec{r}_\vartheta \times \vec{r}_\varphi = (-\cos^2 \vartheta \cos \varphi, \cos^2 \vartheta \sin \varphi,$$

$$-\sin \vartheta \cos \vartheta \cos \varphi - \cos \vartheta \sin \vartheta \sin \varphi) =$$

$$= (-\cos^2 \vartheta \cos \varphi, \cos^2 \vartheta \sin \varphi, -\sin \vartheta \cos \vartheta) =$$

$$= -\cos \vartheta \underbrace{(\cos \vartheta \cos \varphi, -\cos \vartheta \sin \varphi, \sin \vartheta)}_{\vec{N}}$$

$$\int_S \vec{f} \, d\vec{S} = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} (a \cos^2 \vartheta \cos \varphi + b \cos^2 \vartheta \sin \varphi - c \sin \vartheta \cos \vartheta) \, d\vartheta$$

$$= \int_0^{\frac{\pi}{2}} \left((-a \cos \varphi + b \sin \varphi) \cos^2 \vartheta \cdot \frac{1}{2} \sin 2\vartheta \right) d\vartheta =$$

Pravedemo na Bernoulli

$$= \int_0^{\frac{\pi}{2}} (-a \cos \varphi + b \sin \varphi) \left(\frac{\pi}{4} - \frac{c}{2} \right) d\varphi =$$

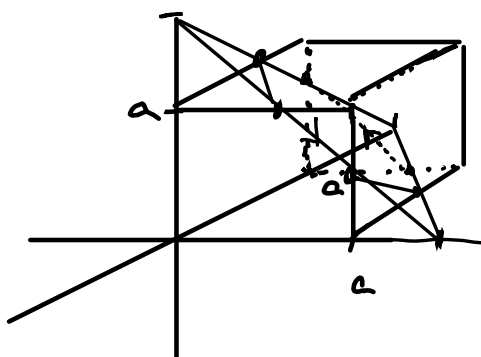
$$= \left(-\frac{\pi a}{4} \sin \varphi + \frac{\pi b}{4} \cos \varphi - \frac{c}{2} \varphi \right) \Big|_0^{\frac{\pi}{2}} =$$

$$= -\frac{\pi}{4} a - \frac{\pi}{4} b - \frac{\pi}{4} c = -\frac{\pi}{4} (a + b + c)$$

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$$\vec{f}(x,y,z) = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$

Izračunaj cirkulacijo f vzdolž preseka roba kocke $[0,a]^3$ in ravnine $x+y+z = \frac{3a}{2}$



kivulj 6 kotnik

cirkulacija... integral vektorskega polja vzdolž sklenjene krivulje

K_1

$$x = x$$

$$y = 0$$

$$z = \frac{3a}{2} - x$$

$$\vec{r}(x) = (x, 0, \frac{3a}{2} - x)$$

$$\int_{K_1} \vec{f} d\vec{r}$$

$$\vec{f}(x) = (1, 0, -1)$$

$$\vec{f}(\vec{r}(x)) = (0 - (\frac{3a}{2} - x)^2, (\frac{3a}{2} - x)^2 - x^2, x^2 - 0)$$

$$\vec{f} \cdot \vec{r}' = -(\frac{3a}{2} - x)^2 - x^2 = -2x^2 + 3ax - \frac{9}{4}a^2$$

$$\int_{\frac{a}{2}}^a (-(\frac{3a}{2} - x)^2 - x^2) dx =$$

$$= -\frac{1}{3} (\frac{3a}{2} - x)^3 \Big|_{\frac{a}{2}}^a - \frac{1}{3} x^3 \Big|_{\frac{a}{2}}^a =$$

$$= -\frac{1}{3} \frac{a^3}{8} + \frac{1}{3} a^3 - \frac{1}{3} a^3 + \frac{1}{3} \frac{a^3}{8} = 0$$

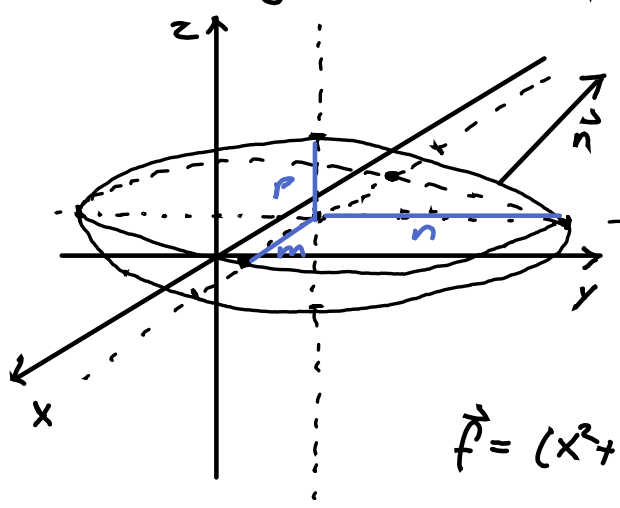
$$m, n, p > 0$$

$$a, b, c \in \mathbb{R}$$

$$I = \int_S x^2 dz dy + y^2 dx dz + z^2 dx dy$$

$$S: \left(\frac{x-a}{m}\right)^2 + \left(\frac{y-b}{n}\right)^2 + \left(\frac{z-c}{p}\right)^2 = 1$$

zunanje stran te ploskve



$$\vec{f} = (x^2 + y^2 + z^2)$$

$$I = \int_S \vec{f} dS = \int_D \text{div} \vec{f} dV \quad \text{gaussov : zrek}$$

Normala mora biti
zunanje

$$\int_D (2x + 2y + 2z) dV = 2x_T + 2y_T + 2z_T$$

$$x_T = \int_D x dV = x_T \cdot V(D) = x_T V(D)$$

$$\text{enotske kugle } B: V(B) = \frac{4}{3} \pi r^3 = \frac{4}{3} \pi$$



$$V(D) = m \cdot n \cdot p \cdot \frac{4}{3} \pi$$

D dobimo če B raztegujemo o faktor
m v smeri x, n, p v drugih dveh

B opisemo:

$$\begin{aligned} x &= r \cos \vartheta \cos \varphi \\ y &= r \cos \vartheta \sin \varphi \\ z &= r \sin \vartheta \end{aligned} \quad r \in [0, 1]$$

D opisemo:

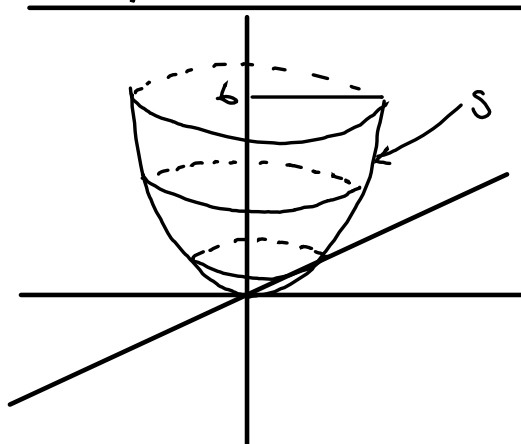
$$\begin{aligned} x &= m r \cos \vartheta \cos \varphi \\ y &= n r \cos \vartheta \sin \varphi \\ z &= p r \sin \vartheta \end{aligned}$$

$$I = 2 \frac{4}{3} \pi m \cdot n \cdot p (a + b + c)$$

$\vec{f}(\vec{r}) = |\vec{r}|^2 \vec{r} \quad b > 0$ ~~////////////////////~~
 izračunaj pretok $\vec{f}$ skozi

a) rob območja $D = \{x, y, z; 2z \geq x^2 + y^2; z \leq b\}$

b) ploskev $2z = x^2 + y^2; z \leq b$



Pretok skozi S :

$$\int_S \vec{f} d\vec{s}$$

$$\int_{\partial D} \vec{f} d\vec{s} = \int_D \operatorname{div} \vec{f} dV$$

$$\begin{aligned} \vec{f}(x, y, z) &= (x^2 + y^2 + z^2)(x, y, z) = \\ &= (x(x^2 + y^2 + z^2), y(x^2 + y^2 + z^2), z(x^2 + y^2 + z^2)) = \\ &= (x^3 + yx^2 + z^2x, xy^3 + y^3 + z^2y, x^2z + y^2z + z^3) \end{aligned}$$

$$\begin{aligned} \operatorname{div} \vec{f} &= 3x^2 + y^2 + z^2 + x^2 + 3y^2 + z^2 + y^2 + x^2 + 3z^2 \\ &= 5x^2 + 5y^2 + 5z^2 \end{aligned}$$

$$\int_D \operatorname{div} \vec{f} dV = 5 \int_D (x^2 + y^2 + z^2) dV$$

$$\begin{aligned} x &= r \cos \varphi & r^2 &= x^2 + y^2 = 2z \\ y &= r \sin \varphi & z &= \frac{r^2}{2} \\ z &= z \end{aligned}$$

$$= 5 \int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} dr \int_{\frac{r^2}{2}}^b (r^2 + z^2) r dz$$

$$= 5 \int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} \left(r^3 z + \frac{1}{3} z^3 \right) \Big|_{\frac{r^2}{2}}^b dr$$

$$= 10\pi \int_0^{\sqrt{2b}} \left(br^3 + \frac{1}{3} b^3 - \frac{r^5}{2} - \frac{1}{6} r^6 \right) dr =$$

$$= 10\pi \left(\frac{1}{4} br^4 + \frac{1}{3} b^3 r - \frac{1}{12} r^6 - \frac{1}{6 \cdot 7} r^7 \right) \Big|_0^{\sqrt{2b}} =$$

$$= 10\pi \left(\frac{1}{4} b^3 \cdot 4 + \frac{1}{3} b^3 \sqrt{2b} - \frac{1}{12} \cdot 8 b^3 - \frac{1}{6 \cdot 7} \cdot 8 b^3 \sqrt{2b} \right)$$

~~neke~~ *neke*

$$= 10\pi \left(b^3 + \frac{b^4}{3} - \frac{2}{3} b^3 - \frac{b^4}{12} \right) =$$

$$10\pi \left(\frac{b^3}{3} + \frac{b^4}{4} \right)$$

b) $\partial D = S \cup S_0$

$$\int_S \vec{f} d\vec{s} = \int_{\partial D} \vec{f} d\vec{s} - \int_{S_0} \vec{f} d\vec{s}$$

$$\int_{S_0} \vec{f} d\vec{s} = \int_{S_0} (\vec{f} \cdot \vec{n}) d\vec{s} = \quad \vec{n} = (0, 0, 1)$$

$$= \int_{S_0} (2x^2 + 2y^2 + z^3) d\vec{s} = \int_{S_0} (bx^2 + by^2 + b^3) d\vec{s} =$$

$$\begin{aligned} x &= r \cos \varphi \\ y &= r \sin \varphi \end{aligned}$$

$$= b \int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r(r^2 + b^2) dr = b \left(\int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r^3 dr + \int_0^{2\pi} b^2 d\varphi \right)$$

$$= b \left(2\pi b^3 + \int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r^3 dr \right) =$$

$P(S_0) = 2\pi b$

$$= b \left(2\pi b^3 + 2\pi \cdot \frac{1}{4} \cdot 4b^2 \right) =$$

$$= 2\pi b^4 + 2\pi b^3$$

$$\Rightarrow \int_S \vec{f} d\vec{s} = \frac{10}{3} \pi b^3 + \frac{10}{3} \pi b^4 - 2\pi b^4 - 2\pi b^3$$

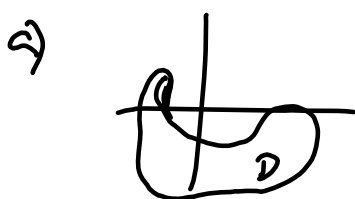
trazimoj $I = \int_K \frac{x dy - y dx}{x^2 + y^2}$

cc je $K \subseteq \mathbb{R}^2$ sklenjena krivulja;

a) ki ne obkroži izhodišča

b) ki dohodi izhodišče

Omejimo se na primer: $K = \partial D$ za D^2 odsekoma gladkim robom



$$\int_{\partial D} P dx + Q dy = \int_D (Q_x - P_y) dx dy$$

↑
pozitivna orientacija

↑
greenova formula

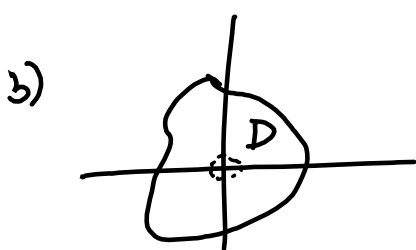
$$P = \frac{-y}{x^2 + y^2} \quad Q = \frac{x}{x^2 + y^2}$$

$$Q_x = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$P_y = \frac{-x^2 + y^2}{(x^2 + y^2)^2}$$

$$\int_D (Q_x - P_y) dx dy = \int_D 0 dx dy = 0$$

∂D ne obkroži $(0,0) \Leftrightarrow (0,0) \notin D$
 $(x^2 + y^2) \neq 0$



$K(0, \epsilon)$ je D rezano dovolj majhen krog

$$D' = D - K(0, \epsilon)$$

Uporabimo Greenovo formulo na D'

$$\int_{\partial D'} P dx + Q dy = \int_{\partial D'} (Q_x - P_y) dx dy = 0$$

$$\partial D' = \partial D^+ - \partial K(0, \epsilon)^-$$

$$0 = \int_{\partial D^+} P dx + Q dy + \int_{\partial K^-} P dx + Q dy$$

Pozor orientacije obratna kot ponavadi

$$\Rightarrow \int_{\partial D^+} \dots = \int_{\partial K^+} \dots$$

$$x = \epsilon \cos \varphi \quad y = \epsilon \sin \varphi$$

$$\frac{dx}{d\varphi} = -\epsilon \sin \varphi$$

$$\frac{dy}{d\varphi} = \epsilon \cos \varphi$$

$$\int_{\partial K} \dots = \frac{1}{\epsilon} \int_0^{2\pi} \cos^2 \varphi + \sin^2 \varphi = 2\pi$$

$r_1, \dots, r_n \in \mathbb{R}^3$ različiti

$e_1, \dots, e_n \in \mathbb{R}$

$$f(\vec{r}) = \sum_{i=1}^n \text{grad} \left(\frac{e_i}{4\pi |\vec{r} - \vec{r}_i|} \right)$$

iracionalnej pretok $\vec{f}$ skozi elipsoid

plaster s k. odprane kake $r_1, \dots, r_n$



$r_1, \dots, r_n \in \text{int}(D)$ $S = \partial D$

$$\int_{\partial D} \vec{f} d\vec{S} = \int_D \text{div} \vec{f} dV$$

↑ D

gaussov izrek vjetno ne velja,

ker ima $\vec{f}$ singularne točke v D

$$\text{div}(\vec{f}(\vec{r})) = \sum_{i=1}^n \text{div} \left(\text{grad} \left(\frac{e_i}{4\pi |\vec{r} - \vec{r}_i|} \right) \right)$$

Spomimo se: $\text{grad} \frac{1}{|\vec{r} - \vec{a}|} = -\frac{\vec{r} - \vec{a}}{|\vec{r} - \vec{a}|^3}$

$$\text{in } \text{div} \left(\text{grad} \frac{1}{|\vec{r} - \vec{a}|} \right) = 0$$

Iz D izrežemo kugle $K_i = K(\vec{r}_i, \epsilon)$

Tako da $\bar{K}_i \subset \text{int}(D)$ in $K_i \cap K_j = \emptyset$ za $i \neq j$

$$D' = D - \left(\bigcup_{i=1}^n K_i \right)$$

Na D' uporabimo gaussov izrek

$$\int_{\partial D'} \vec{f}(\vec{r}) d\vec{S} = \int_{D'} \text{div} \vec{f} dV = 0$$

$$\partial D' = \partial D^+ \cup \left(\bigcup_{j=1}^n \partial K_j^- \right) \Rightarrow \int_{\partial D'} = \int_{\partial D^+} + \sum_{i=1}^n \int_{\partial K_i^-} = 0$$

$$\Rightarrow \int_{\partial D^+} = - \sum_{i=1}^n \int_{\partial K_i^-}$$

$$\int_{\partial K_i^-} \vec{f} d\vec{S} = \int_{K_i} \sum_{j=1}^n \text{grad} \left(\frac{e_j}{4\pi |\vec{r} - \vec{r}_j|} \right) dV$$

$$\sum_{j=1}^n \int_{\partial K_i^-} \left(\frac{e_j}{4\pi} \frac{\vec{r} - \vec{r}_j}{|\vec{r} - \vec{r}_j|^3} \right) d\vec{S}$$

$$\int_{\partial K_i^-} \frac{\vec{r} - \vec{r}_j}{|\vec{r} - \vec{r}_j|^3} d\vec{S} \xrightarrow{i \neq j} \vec{r}_j \in \bar{K}_i \Rightarrow$$

lahko uporabimo
gaussov izrek

$$\Rightarrow = 0$$

$$\sum_{j=1}^n \int_{\partial K_i^-} \frac{e_j}{4\pi} \frac{\vec{r} - \vec{r}_j}{|\vec{r} - \vec{r}_j|^3} d\vec{S} = \frac{e_i}{4\pi} \int_{\partial K_i^-} \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3} d\vec{S}$$

$$d\vec{S} = \vec{n} dS$$



$$\vec{n} = \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|}$$

$$= \frac{e_i}{4\pi} \int_{\partial K_i^-} \frac{|\vec{r} - \vec{r}_i|^2}{|\vec{r} - \vec{r}_i|^4} dS = \frac{e_i}{4\pi} \int_{\partial K_i^-} \frac{dS}{|\vec{r} - \vec{r}_i|^2} =$$

na ∂K_i^- je
 $|\vec{r} - \vec{r}_i| = \epsilon$

$$= \frac{e_i}{4\pi} \int \frac{dS}{\epsilon^2} =$$

$$\frac{e_i}{4\pi \epsilon^2} P(\partial K_i^-) = \frac{e_i}{4\pi \epsilon^2} 4\pi \epsilon^2 = e_i$$

$$\Rightarrow \int \vec{f} d\vec{S} = \sum_{i=1}^n e_i$$

$$I = \int_S (1+x^2) f(x) dy dz - 2xy f(x) dz dx - 3z f(x) dx dy$$

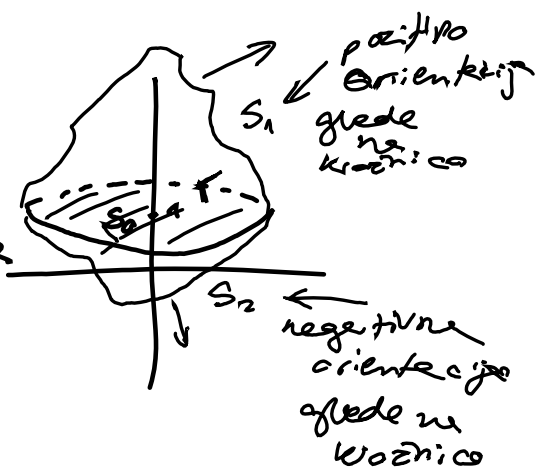
Dalo: $f \in C^1$ da I enak ~~na~~ vse ploskve S , katerih rob je krožnica $\{(\cos t, \sin t, 1) \}$
 Dokaži to ravnost I

$$\vec{F} = ((1+x^2)f(x), 2xy f(x), -3z)$$

$$\Rightarrow I = \int_S \vec{F} d\vec{S}$$

$S_1 \cup S_2$ je sklenjena ploskev

$$S_1^+ \cup S_2^- = \partial D^+$$



Gauss na D :

$$\int_{\partial D} \vec{F} d\vec{S} = \int_D \operatorname{div} \vec{F} dV$$

$$\partial D = \int_{S_1} - \int_{S_2} = 0 \iff \operatorname{div} \vec{F} dV = 0$$

$\vec{F}$ je solenoidalno ($\operatorname{div} \vec{F} = 0$) $\Rightarrow f(x)$ je oblike $3 \arctan x + c$

$$\int_S \vec{F} d\vec{S} = \int_S ((1+x^2)(3 \arctan x + c), -2xy(3 \arctan x + c), -3z) d\vec{S}$$

$$= \int_{S_0} \vec{F} d\vec{S} =$$

na krogu: $d\vec{S} = \vec{n} \cdot dS$ $\vec{n} = (0, 0, 1)$

$$= \int_{S_0} -3z dS = -3 \int_{S_0} z dS = -3z_T P(S_0) =$$

$$= -3 \cdot 1 \cdot \pi$$

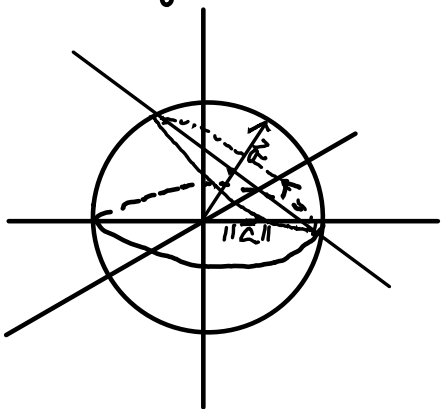
Opomba: Lahko bi dokazali: da je
tudi $\text{div} \vec{f}$ polinomska funkcija

$$\vec{a} \neq \vec{0}$$

$$\vec{b} \in \mathbb{R}$$

$$\vec{F}(\vec{r}) = (\vec{r} - \vec{a}) \times (\vec{r} - \vec{b})$$

izračunaj cirkulacijo $\vec{F}$ vzdolž
kružnice K : $|\vec{r}| = |\vec{a}|$ in $\vec{r} \cdot \vec{a} = \frac{|\vec{a}|^2}{2}$



enačba ravnine:

$$\vec{r} \cdot \vec{n} = d$$

$\vec{n}$ normala

ravnina z normalo $\vec{a}$
in točka $\vec{r} = \frac{\vec{a}}{2}$

K je krožnica

B naj bo krožnica z radijem $K = \partial B$

$$\int_{\partial B} \vec{F} d\vec{r} = \int_B \text{rot } \vec{F} d\vec{s}$$

Stokesov izrek

normala: $B: \vec{n} = \frac{\vec{a}}{|\vec{a}|}$

$$\text{rot } \vec{F} = \text{rot}(\underbrace{\vec{r} \times \vec{r}}_0 - \vec{a} \times \vec{r} - \vec{r} \times \vec{b} + \vec{a} \times \vec{b}) =$$

$$\text{Vemari } \text{rot}(\vec{r} \times \vec{c}) = -2\vec{c}$$

$$= -2\vec{a} + 2\vec{b} + 0 = 2(\vec{b} - \vec{a})$$

$$I = \int_B 2(\vec{b} - \vec{a}) d\vec{s} = \frac{2}{|\vec{a}|} \int_B (\vec{b} - \vec{a}) \cdot \vec{a} d\vec{s} =$$

$$= \frac{2(\vec{b} - \vec{a}) \cdot \vec{b}}{|\vec{a}|} \cdot P(B)$$

$$P(B) = ?$$

polmer:



$$b = \sqrt{|\vec{a}|^2 - \frac{|\vec{a}|^2}{4}} = \frac{\sqrt{3}}{2} |\vec{a}|$$

$$P(B) = \pi b^2 = \pi \cdot \frac{3}{4} |\vec{a}|^2$$

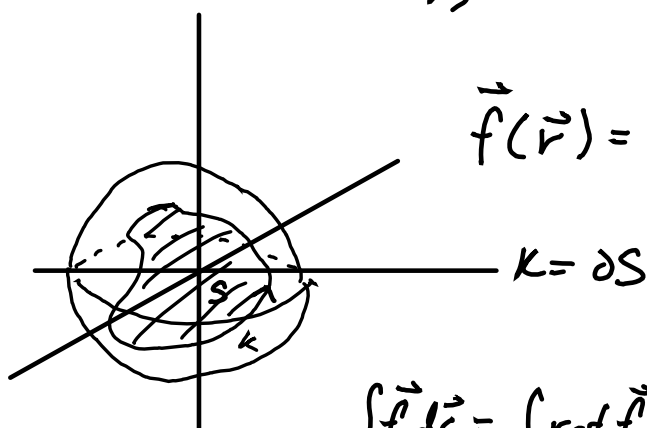
$$I = 3\pi (\vec{b} - \vec{a}) \cdot \vec{b} |\vec{a}|$$

K je isti $\vec{b}$ kot b

K zaključena krogla
na sferi $x^2 + y^2 + z^2 = 1$

Izračunaj $I = \int_K \frac{dx + dy + dz}{(x^2 + y^2 + z^2)^2} = \int_K \vec{f} d\vec{r}$

$$\vec{f} = \left(\frac{1}{(x^2 + y^2 + z^2)^2}, \frac{1}{(x^2 + y^2 + z^2)^2}, \frac{1}{(x^2 + y^2 + z^2)^2} \right)$$



$$\vec{f}(\vec{r}) = (1, 1, 1) \text{ na } \text{res}$$

$$\int_K \vec{f} d\vec{r} = \int_S \text{rot} \vec{f} d\vec{S} = \dots$$

↑
obstaja boljši način

$$\int_K \vec{f} d\vec{r} = \int_K \underbrace{(1, 1, 1)}_{\vec{g}(\vec{r})} d\vec{r} = \int_S \text{rot}(\vec{g}(\vec{r})) d\vec{S} =$$

$$= \int_S 0 d\vec{S} = 0$$

Kaj če $K \neq \partial S$

če je poje $\vec{f}$ potencialno ($\vec{f} = \text{grad } u$)
in K krivulje z začetkom a in koncem
 $b \Rightarrow \int_K \vec{f} d\vec{r} = u(b) - u(a)$

Poizvedica: če je K sklenjena je integral
vedel $\vec{0}$ od $\int_K \text{grad } u d\vec{r} = \vec{0}$

$\vec{f}$ ni potencialno, ampak $\vec{g}$ je potencialno
(ker $\text{rot} \vec{g} = 0$)

$$\vec{g} = \text{grad}(x + y + z) \quad I = \int_K \vec{g} d\vec{r} = 0$$

Opomba: $\int_K \ln(x^2 + y^2 + z^2) (dx + dy + dz)$

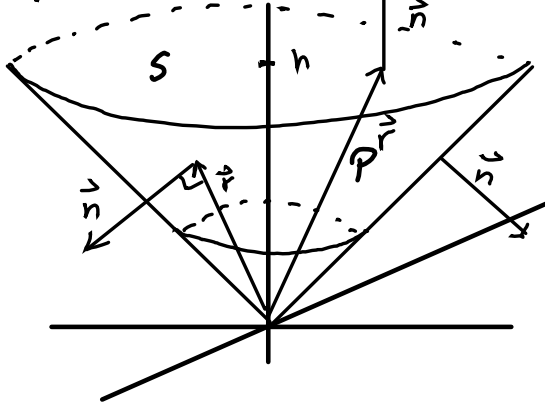
↓ na S ,

konstantno
poje

$$h > 0$$

$$\vec{f}(\vec{r}) = \vec{r}$$

končnemu pretoku $\vec{f}$ skozi ploščo in sredi
ploskev stožca: $x^2 + y^2 \leq z^2$ $z \in [0, h]$



$$\vec{r} \cdot \vec{n} = 0$$

$$\int_P \vec{f} dS = \int_P \vec{f} \vec{n} dS = \int_P \vec{r} \vec{n} dS = \int 0 dS = 0$$

$$\int_S \vec{f} dS = \int_S \vec{r} \vec{n} dS = \int_S z dS = z_T \cdot P(S) =$$

$$\vec{n} = (0, 0, 1) \quad = h \cdot \pi h^2 = \pi h^3$$

2. način (z gaussovo)

$$\int \vec{f} dS = \int_D \text{div} \vec{f} dV = \int_D (1+1+1) dV = 3P(V) =$$

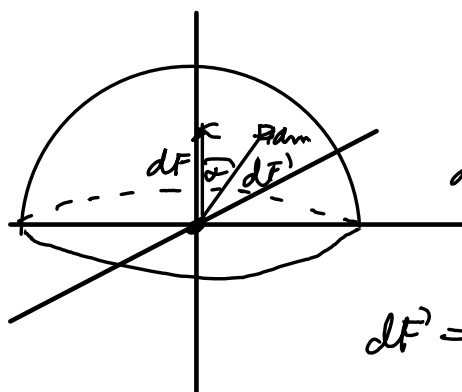
$$= 3 \cdot \frac{\pi h^2 \cdot h}{3} = \pi h^3$$

$$\int_{\partial D} = \int_S + \int_P$$

$$a > 0$$

Dobiti privlačno silo med točkastim telesom z maso m_0 , ki je v izhodišču, in ploščkijo $x^2 + y^2 + z^2 = a^2$ $a > 0$ z maso M

homogena



$$F = \frac{m_1 m_2}{r^2} G$$

$$dm = \rho dS$$

$$dF = \frac{m_0 dm}{r^2} G = \frac{m_0 \rho dS}{r^2} G$$

Rezultante bo kerala agr torej nes
zanima samo z komponente

$$dF = dF \cos \alpha$$

$$x = a \cos \vartheta \cos \varphi$$

$$\alpha = \frac{\pi}{2} - \vartheta$$

$$y = a \cos \vartheta \sin \varphi$$

$$z = a \sin \vartheta$$

$$dF = \frac{G m_0 \rho}{r^2} \cos \left(\frac{\pi}{2} - \vartheta \right) dS$$

$$dS = a^2 \cos \vartheta d\varphi d\vartheta$$

$$\text{ok prej } \sqrt{EG - F^2} = a^2 \cos \vartheta$$

$$F = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \frac{G m_0 \rho}{a^2} \sin \vartheta a^2 \cos \vartheta d\vartheta =$$

$$= G m_0 \rho 2\pi \int_0^{\frac{\pi}{2}} \sin \vartheta \cos \vartheta d\vartheta =$$

$$G m_0 \rho \pi \int_0^{\frac{\pi}{2}} \sin 2\vartheta d\vartheta = \frac{\pi}{2} G m_0 \rho \cos 2\vartheta \Big|_0^{\frac{\pi}{2}} = \oplus$$

$$= \pi G m_0 \rho = \frac{G m_0 \cdot \pi}{2 a^2}$$

$$\uparrow \rho = \frac{M}{P(S)} = \frac{M}{2\pi a^2}$$

$D \subseteq \mathbb{R}^3$ gaussesto območje (območje z gladlim robom) in prostornino V

$$a \in \mathbb{R}^3$$

Izračunaj:

$$\int_{\partial D} (\vec{r} \times \vec{a}) \times d\vec{s} = I$$

$$(\vec{x} \times \vec{y}) \times \vec{z} = (\vec{x} \cdot \vec{z}) \vec{y} - (\vec{y} \cdot \vec{z}) \vec{x}$$

$$\vec{I} = \int_{\partial D} (\vec{r} \cdot d\vec{s}) \vec{a} - (\vec{a} \cdot d\vec{s}) \vec{r} =$$

$$d\vec{s} = \vec{n} \cdot dS$$

$$= \int_{\partial D} ((\vec{r} \cdot \vec{n}) \vec{a} - (\vec{a} \cdot \vec{n}) \vec{r}) dS =$$

$$a = (a_1, a_2, a_3)$$

$$r = (x, y, z)$$

$$= \int_{\partial D} ((\vec{r} \cdot \vec{n}) dS) (a_1, a_2, a_3) - (\vec{a} \cdot \vec{n} x, \vec{a} \cdot \vec{n} y, \vec{a} \cdot \vec{n} z) dS$$

$$= \left(\int_{\partial D} (\vec{r} \cdot \vec{n}) dS \right) \vec{a} - \underbrace{\int_{\partial D} (\vec{a} \cdot \vec{n} x, \vec{a} \cdot \vec{n} y, \vec{a} \cdot \vec{n} z) dS}_{\int_{\partial D} x \vec{a} \cdot \vec{n} dS}$$

$$\rightarrow \left(\int_{\partial D} x \vec{a} \cdot \vec{n} dS, \int_{\partial D} y \vec{a} \cdot \vec{n} dS, \int_{\partial D} z \vec{a} \cdot \vec{n} dS \right)$$

$$\int_{\partial D} (\vec{r} \cdot \vec{n}) dS = \int_{\partial D} \vec{r} \cdot d\vec{s} = \int_D \operatorname{div} \vec{r} dV =$$

$$= 3 \int_D dV = 3V$$

$$\int_{\partial D} x \vec{a} \cdot \vec{n} dS = \int_{\partial D} x \vec{a} \cdot d\vec{s} = \int_D \operatorname{div} (x \vec{a}) dV = a_1 V$$

$$\operatorname{div} x \vec{a} = \nabla (x a_1, x a_2, x a_3) = a_1$$

Podobno y in

$$I = 3V \vec{a} - (a_1 V, a_2 V, a_3 V) = 2 \vec{a} V$$

Dokazati da je za

$$\vec{f} = (2x \cos y - y^2 \sin x, 2y \cos x - x^2 \sin y, u)$$

$\int_K \vec{f} d\vec{r}$ enak za vse krivulje med
 $(0,0,0)$ in $(5,3,\pi)$
in izračunaj

$$\int_{K_0} \vec{f} d\vec{r} \quad \text{za } K_0: \vec{r}(t) = (\cos t, \sin t, t); \quad t \in [-2\pi, 2\pi]$$

Dokazujemo: $\vec{f}$ je potencialno

(smo že videli na začetku)

$$u = x^2 \cos y + y^2 \cos x + uz$$

↑
potential za $\vec{f}$

$$\int_{K_0} \vec{f} d\vec{r} = u(b) - u(a) = u(1, 0, 2\pi) - u(1, 0, -2\pi)$$

$$= 4 \cdot 2\pi - (-4 \cdot 2\pi) = 16\pi$$

$$I = \int_K u(x, y) (y dx + x dy)$$

$$\vec{f}(x, y) = (y, x)$$

$$\vec{g} = u \cdot \vec{f} = u(x, y) \cdot (y, x)$$

Radi bi da je $\vec{g}$ potencijalno polje

$$\vec{g} = \text{grad } v = (v_x, v_y)$$

$$v_x = u(x, y) \cdot y \quad v_y = u(x, y) \cdot x$$

če odvajamo

eno po x eno po y mera pitajmo

$$v_{xy} = u(x, y) + u_y y \quad v_{yx} = u_x x + u(x, y)$$

$$\leadsto y u_y = x u_x$$

• Ta prajj lahko dobimo z greenom

• vložimo v ravnino v $\mathbb{R}^3$

$$t = x \cdot y$$

$$s = \frac{x}{y}$$

$$I = \int_K y u dx + x u dy + 0 dz$$

$$\vec{h} = (y u, x u, 0) \text{ je potencijalno} \Leftrightarrow \text{rot } \vec{h} = 0$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial t} \frac{\partial t}{\partial x} + \frac{\partial}{\partial s} \frac{\partial s}{\partial x} = y \frac{\partial}{\partial t} + \frac{1}{y} \frac{\partial}{\partial s}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial t} \frac{\partial t}{\partial y} + \frac{\partial}{\partial s} \frac{\partial s}{\partial y} = x \frac{\partial}{\partial t} - \frac{x}{y^2} \frac{\partial}{\partial s}$$

$$x(y u_t + \frac{1}{y} u_s) - y(x u_t - \frac{x}{y^2} u_s) = 0$$

$$t u_t + s u_s - t u_t + s u_s = 2 s u_s \quad s > 0$$

$$u_s = 0 \Rightarrow u = u(t) \dots \text{samo od } t$$

$$u = h(t)$$

adizem

$$u = h(x \cdot y) \quad \text{kjer } h \in C^1$$

$u, l \in C^2$ $D \subseteq \mathbb{R}^3$ z odsekom a
gladkim robom

$\vec{e} = \mathbb{R}^3$ enotski:

$$\text{Smerni odvod: } \frac{\partial u}{\partial \vec{e}} = \text{grad} u \cdot \vec{e}$$

Opomba: $\frac{\partial u}{\partial (1,0,0)} = \frac{\partial u}{\partial x} \cdot \vec{e}$

$\vec{n}$... zunanja enotska normala za ∂D

a) Dokaži:

$$\int_{\partial D} u \cdot \frac{\partial v}{\partial \vec{n}} dS = \int_D \text{grad} u \cdot \text{grad} v + u \nabla v \, dv$$

$$b) \int_{\partial D} \left(u \frac{dv}{\partial \vec{n}} - v \frac{du}{\partial \vec{n}} \right) dS = \int_D (u \nabla v - v \nabla u) \, dv$$

$$\int_{\partial D} u \frac{dv}{\partial \vec{n}} dS = \int_{\partial D} u \text{grad} v \cdot \underbrace{\vec{n}}_{\frac{d\vec{s}}{dS}} dS = \int_{\partial D} \underbrace{\text{grad} u}_{\vec{f}} \cdot d\vec{s}$$

$$\text{div } \vec{f} = \text{div}(u \text{grad} v) = \text{div}(u v_x, u v_y, u v_z) =$$

$$= (u v_x)_x + (u v_y)_y + (u v_z)_z =$$

ko razpišemo

$$= \text{grad} u \cdot \text{grad} v + u \nabla v$$

b)

$$\int_{\partial D} \left(u \frac{dv}{\partial \vec{n}} - v \frac{du}{\partial \vec{n}} \right) dS = \int_{\partial D} u \frac{dv}{\partial \vec{n}} dS - \int_{\partial D} v \frac{du}{\partial \vec{n}} dS =$$

$$= \int_D \text{grad} u \cdot \text{grad} v + u \nabla v - \text{grad} v \cdot \text{grad} u - v \nabla u \, dv$$

$$f: \mathbb{R}^3 \rightarrow \mathbb{R} \text{ harmonična } (\Delta f = 0)$$

Dokaži da za $a > 0$ velja:

$$\frac{1}{4\pi a^2} \int_{S(0,a)} f dS = f(0,0,0) = I(a)$$

Nasvet: Parametriziraj $S(0,a)$ in nato s pomočjo odvajanja in gaussovega izreka dokeži trditev.

$$x = a \cos \vartheta \cos \varphi \quad \vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$y = a \cos \vartheta \sin \varphi \quad \varphi \in [0, 2\pi]$$

$$z = a \sin \vartheta \quad \leadsto \vec{r}(\varphi, \vartheta)$$

$$dS = |\vec{r}_\varphi \times \vec{r}_\vartheta| d\varphi d\vartheta = a^2 \cos \vartheta d\varphi d\vartheta$$

$$I(a) = \frac{1}{4\pi a^2} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2 \cos \vartheta \underbrace{f(\vec{r}(\varphi, \vartheta))}_{g(a, \varphi, \vartheta)} d\vartheta$$

$$I'(a) = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial g}{\partial a} d\vartheta =$$

$$= \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \vartheta (f_x(\vec{r}(\varphi, \vartheta)) \cdot x_a + f_y(\vec{r}(\varphi, \vartheta)) \cdot y_a + f_z(\vec{r}(\varphi, \vartheta)) \cdot z_a) d\vartheta =$$

$$f(x, y, z) = f(a \cos \vartheta \cos \varphi, a \cos \vartheta \sin \varphi, a \sin \vartheta)$$

$\forall \vec{r}$ se shriva a

$$= \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \vartheta (f_x \cos \varphi \cos \vartheta + f_y \sin \varphi \cos \vartheta + f_z \sin \vartheta) d\vartheta$$

komponente enotske
normalne $\vec{n}$ na $S(0,1)$

$$= \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \vartheta \underbrace{\text{grad } f \cdot \vec{n}}_{dS \text{ za } a=1} d\vartheta = \quad dS = a^2 \cos \vartheta d\vartheta d\varphi$$

$$= \frac{1}{4\pi} \int_{S(0,1)} \underbrace{\text{grad } f}_{\Delta f} \cdot \vec{n} \cdot d\vec{S} = \frac{1}{4\pi} \int_{S(0,1)} \text{grad } f \cdot d\vec{S} =$$

$$= \frac{1}{4\pi} \int_{K(0,1)} \text{div}(\text{grad } f) dV = 0$$

$$\Delta f = f_{xx} + f_{yy} + f_{zz}$$

$I'(a)$ je 0 $\Rightarrow I(a)$ je konstantna

$$I(a) = \frac{1}{4\pi a^2} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2 \cos \vartheta f(\vec{r}(\varphi, \vartheta)) d\vartheta$$

$$\lim_{a \rightarrow 0} I(a) = \frac{1}{4\pi} \lim_{a \rightarrow 0} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \vartheta f(\vec{r}(\varphi, \vartheta)) d\vartheta =$$

$$= \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \lim_{a \rightarrow 0} \cos \vartheta f(\vec{r}(\varphi, \vartheta)) d\vartheta =$$

$$= \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \vartheta f(0,0,0) d\vartheta =$$

$$= \frac{1}{4\pi} f(0,0,0) \cdot 4\pi = f(0,0,0) \quad \leftarrow P(S(0,1)) \quad \text{ker je } \cos \vartheta \text{ determinanta}$$

Holomorfne funkcije

$D \subseteq \mathbb{C}$ oblast
 koje od f_1, f_2 i f_3 so holomorfne

$$f_1(z) = \overline{f(z)}$$

$$f_2(z) = f(\bar{z})$$

$$f_3(z) = \overline{f(\bar{z})}$$

$$\text{holomorfna, } \forall a \in D. \exists \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} =: f'(a)$$

$$f(x+iy) = u(x,y) + i v(x,y) \quad u = \operatorname{Re} f \quad v = \operatorname{Im} f$$

$$f \in O(D) \iff u_x = v_y \wedge u_y = -v_x$$

$u, v \in C^1(D)$

$$f_1(x+iy) = \underbrace{u(x,y)}_{u_1(x,y)} - i \underbrace{v(x,y)}_{v_1(x,y)}$$

$$\begin{array}{ll} u_{1x} = u_x & u_{1y} = u_y \\ v_{1x} = -v_x & v_{1y} = -v_y \\ u_{1x} = u_x = v_y = -v_{1y} \\ u_{1y} = -v_y \end{array}$$

$$\Rightarrow \begin{array}{l} v_y = -v_y \\ v_x = -v_x \end{array} \text{ in } \dots$$

$\Downarrow$

$$\begin{array}{l} v_x = 0 = v_y \\ u_x = u_y = 0 \end{array}$$

? ? ?

u, v konstante;

f_1 ni holomorfne

$$f_2(x+yi) = u_2(x,y) + i v_2(x,y); \quad \left. \begin{array}{l} f_2 \text{ ne} \\ \overline{D} \end{array} \right\}$$

$$\parallel$$

$$u(x, -y) + i v(x, -y)$$

~~$$u_{2x} = u_x \quad u_{2y} = u_y \cdot (-1) = -u_y$$

$$v_{2x} = v_x \quad v_{2y} = -v_y$$~~

$$u_{2x}(x,y) = u_x(x, -y) \quad u_{2y}(x,y) = -u_y(x, -y)$$

$$v_{2x}(x,y) = v_x(x, -y) \quad v_{2y}(x,y) = -v_y(x, -y)$$

$$u_{2x}(x,y) = u_x(x, -y) = v_y(x, -y) = -v_{2y}(x,y)$$

Torej f_2 ni holomorfe, razen če so ~~u, v~~^{u, v} konstanti

$$f_3(x+yi) = u_3(x,y) + i v(x,y) \Rightarrow$$

$$u_3(x,y) = u(x, -y)$$

$$v_3(x,y) = -v(x, -y)$$

$$(u_3)_x(x,y) = u_x(x, -y) \quad v_{3x}(x,y) = v_x(x,y)$$

$$(u_3)_y(x,y) = -u_y(x, -y) \quad v_{3y}(x,y) = +v_y(x, -y)$$

$$(u_3)_x = (v_3)_y \Leftrightarrow$$

$$u_x(x, -y) = +v_y(x, -y) \Leftrightarrow u_x = v_y \quad \checkmark$$

$$u_{3y} = -v_{3x} \Leftrightarrow$$

$$-u_y(x, -y) = -v_x(x, -y) \Leftrightarrow v_y = u_x \quad \dots$$

je holomorfe

Dokaži da obstaj $f \in O(\mathbb{C})$ za katero
je $u(x,y) = x^3 - 3xy^2$ in jo določi

$$u_x = 3x^2 - 3y^2 = v_y$$

$$u_y = -6xy = -v_x$$

$$v = 3 \int (x^2 - y^2) dy = 3x^2y - y^3 + C(x)$$

$$v = 6 \int xy dx = 3x^2y + D(y)$$

$$v = 3x^2y - y^3 + C$$

"unija členov"

$$f(x+yi) = x^3 - 3xy^2 + i(3x^2y - y^3 + C)$$

$$f(z) = z^3 + iC \leftarrow \text{z ugibanjem}$$

Brez navdih:

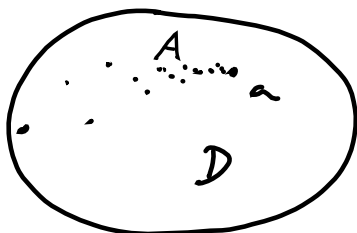
$$z = x + yi \Rightarrow x = \frac{z + \bar{z}}{2} \quad y = \frac{z - \bar{z}}{2i}$$

DN: ugotovi natri

Imenitna trditvi:

$D \subseteq \mathbb{C}$ neko območje, $f, g \in \mathcal{O}(D)$

funkeji ki se ujemata na množici A s
stekatiščem v D



v A obstaja zaporedje
 $(a_n)_n$ $a_n \in A$ $a_n \neq a$
 $\lim_{n \rightarrow \infty} a_n = a$

$$\Rightarrow f = g$$

$$u = x^3 - 3xy^2 \quad v = 3x^2y - y^3 + iC$$

če najdemo $g \in \mathcal{O}(C)$ ki se z f ujemata
na $\mathbb{R}$, po trditvi T sledi $f = g$

$$x \in \mathbb{R} \quad f(x) = f(x + i0) = u(x, 0) + v(x, 0) = x^3 + iC$$

$$\Rightarrow f(z) = z^3 + iC$$

Še imenitnejši premislek:

$$u(x, y) \rightsquigarrow f = ?$$

$$x \in \mathbb{R} : f(x) = u(x, 0) + i v(x, 0) \xrightarrow{x \rightarrow z} f(z) =$$

$$v(x, 0) = \int_{CRS} v_x(x, 0) dx = - \int u_y(x, 0) dx$$

$$u_y = -6xy$$

$$v(x, 0) = - \int -6x \cdot 0 dx = +C$$

$$\Rightarrow f(x) = x^3 + iC \Rightarrow f(z) = z^3 + iC$$

Podobno

$$u(x, 0) = \int u_x(x, 0) dx = \int v_y(x, 0) dx$$

$$u(x,y) = e^x (\cos(ky) + \sin(ky))$$

Dokazati da je $u = \operatorname{Re} f$ za $f \in \mathcal{O}(\mathbb{C})$

Če je $u = \operatorname{Re} f \Rightarrow$

$$\begin{aligned} u_x &= v_y & u_{xx} &= v_{yx} \\ u_y &= -v_x & u_{yy} &= -v_{xy} \end{aligned} \Rightarrow u_{xx} = -u_{yy}$$

Podobno

$$v_{xx} = -v_{yy}$$

$$\Rightarrow \Delta u = 0$$

$$u_{xx} = e^x (\cos(ky) + \sin(ky)) = u$$

$$u_y = e^x (-k \sin(ky) + k \cos(ky))$$

$$u_{yy} = k^2 e^x (-\cos(ky) - \sin(ky)) = -k^2 u$$

$$\Delta u = e^x (1 - k^2) (\cos(ky) + \sin(ky)) = 0 \quad \forall k, y$$

$$\Rightarrow k = \pm 1$$

$$v(x,0) = - \int u_y(x,0) dx = - \int e^x k dx = -k e^x + C$$

$$f(x) = u(x,0) + i v(x,0) = e^x - i k e^x + C i = e^x (1 - i k) + C i$$

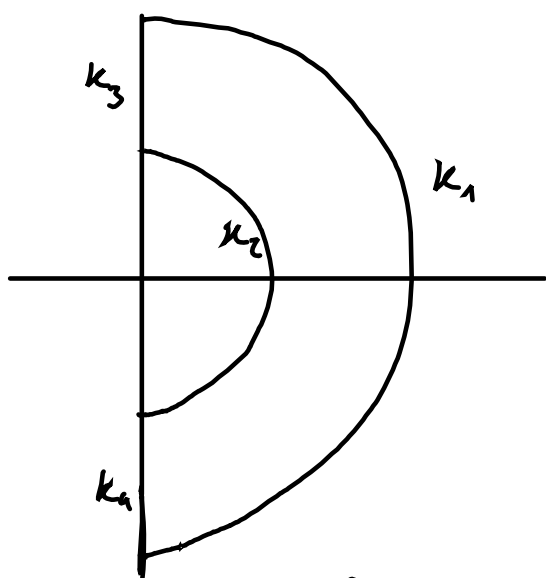
$$f(z) = e^z (1 \pm i) + i C$$



12.4

Integrali holomorfe funkcij

$$f(z) = \frac{z}{\bar{z}} \quad D = \{ z; |z| \in [1, 2] \text{ Re}(z) > 0 \}$$



$$\int_{k_1} f(z) dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{ze^{i\varphi}}{ze^{-i\varphi}} 2i e^{i\varphi} d\varphi =$$

$$z = 2e^{i\varphi}$$

$$dz = 2i e^{i\varphi}$$



$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2i e^{3i\varphi} d\varphi = \frac{2}{3} i e^{3i\varphi} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$\frac{2}{3} i (-i - i) = -\frac{2}{3}; \quad (\text{konje su -})$$

$$\int_{k_2} f(z) dz = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} i \frac{e^{i\varphi}}{e^{-i\varphi}} e^{i\varphi} d\varphi = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{1}{3} i e^{3i\varphi} d\varphi =$$

$$= \frac{1}{3} i (-i + i) = \frac{2}{3}$$

$$\int_{k_3 \cup k_4} f(z) dz = \begin{matrix} z = iy \\ dz = i dy \end{matrix}$$

$$k_3 \cup k_4$$

$$= \int_{-1}^2 \frac{iy}{-iy} i dy + \int_{2-1}^1 \frac{iy}{2-iy} i dy = -i(2+1) + i(1-2) =$$

$$= i(2-1-1+2) \dots = 2i$$

$$f(z) = \bar{z}$$

$D \subseteq \mathbb{C}$ s koso ma gladkim robom

$$\int_{\partial D} f(z) dz = ?$$

$$z = x + iy$$

$$dz = dx + i dy$$

$$I = \int_{\partial D} (x - iy)(dx + i dy) =$$

$$= \int_{\partial D} (x dx + y dy) + i(x dy - y dx) =$$

green

$$= \int_D (0 + 0) + i \int_D (1 - (-1)) dx dy =$$

$$= 2i \cdot P(D)$$

$D \neq \emptyset \subseteq \mathbb{C}$ keine Leerheit

$$f \in \mathcal{O}(D) \cap C^1(\overline{D})$$

$$\begin{aligned} \int_{\partial D} f(z) dz &= \int_{\partial D} (u+iv)(dx+idy) = \\ &= \int_{\partial D} (u dx - v dy) + i(v dx + u dy) = \end{aligned}$$

$$= \int_D (\underbrace{v_x - u_y}_0) dx dy + i \int_D (\underbrace{u_x - v_y}_0) dx dy = 0$$

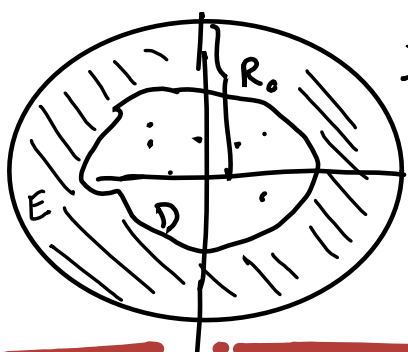
$\left. \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array} \right\} \rightarrow$

$$n \geq 1$$

$z_1, \dots, z_n \in \mathbb{C}$ poljubne med seboj različne

$D \subset \mathbb{C}$ omejeno k vsebuje $z_1, \dots, z_n$

$$\text{Dokaži } \int_{\partial D} \frac{dz}{(z-z_1) \dots (z-z_n)} = 0$$



$$\exists R_0 > 0. \bar{D} \subseteq \Delta(0, R) \quad \forall R > R_0$$

$$\forall R > R_0$$

$$\int_{\partial \Delta(0, R)} f(z) dz = \int_{\partial D} f(z) dz$$

$$E = \Delta(0, R) - \bar{D}$$

f je holomorfna na E

$$\Rightarrow \int_{\partial E} f(z) dz = 0$$

$$\int_{\partial E} f(z) dz = \int_{\partial \Delta(0, R)} f(z) dz - \int_{\partial D} f(z) dz = 0$$

$$\int_{\gamma_R} \frac{dz}{(z-z_1) \dots (z-z_n)} = \int_{\partial D} \frac{dz}{(z-z_1) \dots (z-z_n)}$$

$$\int_{\partial D} \dots = 0$$

$\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz$ če kružni sklopaj
travita E in je f holomorfna na delu E

$$\int_{\gamma_R} |z| = R e^{i\varphi}$$

$$dz = R i e^{i\varphi} d\varphi$$

$$I = \int_0^{2\pi} \frac{i R e^{i\varphi} d\varphi}{(R e^{i\varphi} - z_1) \dots (R e^{i\varphi} - z_n)}$$

$$|I| \leq \int_0^{2\pi} \left| \frac{i R e^{i\varphi}}{(R e^{i\varphi} - z_1) \dots (R e^{i\varphi} - z_n)} \right| d\varphi$$

$$= \int_0^{2\pi} \left| \frac{R}{(R e^{i\varphi} - z_1) \dots (R e^{i\varphi} - z_n)} \right| d\varphi \leq \frac{R}{(R - |z_1|) \dots (R - |z_n|)}$$

$$|R e^{i\varphi} - z_k| \geq |R e^{i\varphi}| - |z_k| = R - |z_k|$$

$$|I| \leq \int_0^{2\pi} \frac{R}{(R - |z_1|) \dots (R - |z_n|)} d\varphi = \frac{2\pi R}{(R - |z_1|) \dots (R - |z_n|)}$$

$$\xrightarrow{R \rightarrow \infty} 0 \Rightarrow I = 0$$

če bimo $n=1$

$$\int_{\partial D} \frac{dz}{z - z_1} \Rightarrow I = 2\pi \text{ ko}$$

$$\int_{\partial D} \frac{f(z) dz}{z - z_0} = 2\pi i f(z_0) \quad f \in \mathcal{O}(D)$$

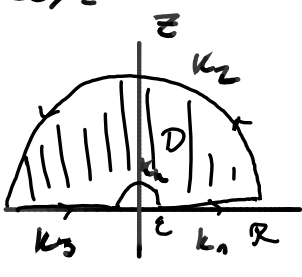
S pomočjo kompleksne integracije 21.4
izračunaj

$$\int_0^{\infty} \frac{\sin x}{x} dx$$

Če imamo $\sin x$ ali $\cos x$ v integralu, potem
je zamenjamo z e^x

Drugi če pa samo zamenjamo x z z

$f(z) = \frac{e^{iz}}{z}$ pomočje



Me moramo
tako, ker gre čez
 $z=0$

R bomo podelili v ∞
in ϵ proti 0

$$\int_{\partial D} f(z) dz = 0$$

$\uparrow$
 $f(z) dz$

∂D razdelimo na ošedke kase:

$$\int_{\partial D} f(z) dz = \int_{k_1} + \int_{k_2} + \int_{k_3} + \int_{k_4}$$

$$k_1: z=x \quad dz=dx \quad x \in [\epsilon, R]$$

$$\begin{aligned} \int_{k_1} f(z) dz &= \int_{\epsilon}^R \frac{e^{ix}}{x} dx = \int_{\epsilon}^R \frac{\cos x + i \sin x}{x} dx \xrightarrow[\epsilon \rightarrow 0^+]{R \rightarrow \infty} = \\ &= \underbrace{\int_0^{\infty} \frac{\cos x}{x} dx}_{\text{ne obstaja}} + i \int_0^{\infty} \frac{\sin x}{x} dx \end{aligned}$$

"Rezultat": združimo k_1 in k_3

$$\int_{[-R, -\epsilon] \cup [\epsilon, R]} \frac{\cos x}{x} dx + i \int_{[-\epsilon, \epsilon] \cup [\epsilon, R]} \frac{\sin x}{x} dx \xrightarrow[\epsilon \rightarrow 0]{R \rightarrow \infty} 2iI$$

$\uparrow$ liha funkcija $\uparrow$ soda funkcija

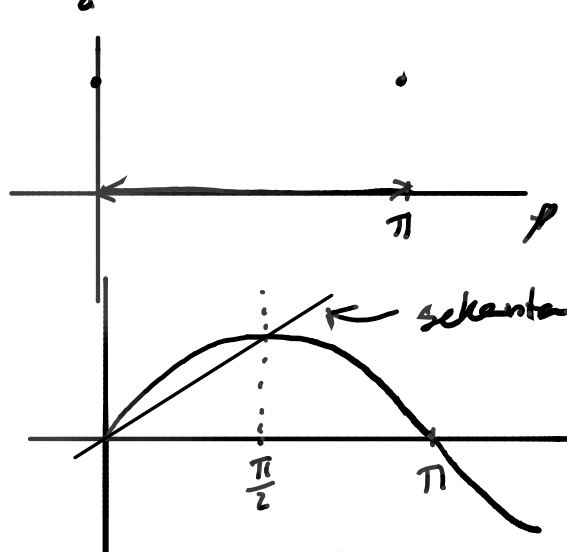
$$k_2: z=Re^{it}$$

$$dz = Re^{it} dt$$

$$\int_{k_2} \dots = \int_0^{\pi} \frac{e^{Re^{it}}}{Re^{it}} Re^{it} dt =$$

$$= i \int_0^{\pi} e^{iRe^{it}} dt = \int_0^{\pi} e^{iR(\cos t + i \sin t)} dt$$

$$\begin{aligned} \left| \int_{k_2} \dots \right| &\leq \int_0^{\pi} |e^{iR(\cos t + i \sin t)}| dt = \\ &= \int_0^{\pi} |e^{-R \sin t} \cdot e^{iR \cos t}| dt = \int_0^{\pi} e^{-R \sin t} dt \xrightarrow{R \rightarrow \infty} 0 \end{aligned}$$



$$\int_0^{\pi} e^{-R \sin t} dt = 2 \int_0^{\pi/2} e^{-R \sin t} dt$$

$$e^{-R \sin t} \leq \sin t \in [0, 1] \quad \text{za } t \in [0, \pi/2]$$

$$-R \sin t \leq 0$$

$$e^{-R \sin t} \leq 1 \quad \therefore$$

$$\sin t \geq \frac{2t}{\pi}$$

$$\int_0^{\pi/2} e^{-R \sin t} dt \leq \int_0^{\pi/2} e^{-\frac{2Rt}{\pi}} dt =$$

$$= -2 \frac{\pi}{2R} (e^{-R} - 1) = \frac{\pi}{R} - \frac{\pi}{2R} e^{-R} =$$

$$\frac{\pi}{R} (1 - e^{-R}) \xrightarrow{R \rightarrow \infty} 0$$

$$\int_{k_4} : z = \epsilon e^{it}$$

...

isto kot pri k_2

$$\int_{k_4} = i \int_{\pi}^0 e^{i\epsilon(\cos t + i \sin t)} dt \xrightarrow[\epsilon \rightarrow 0]{\epsilon \rightarrow 0} i \int_{\pi}^0 1 dt = -i\pi$$

h. am. z: $\int_0^{\pi} g(\epsilon, t) dt \xrightarrow{\epsilon \rightarrow 0} \int_0^{\pi} g(0, t) dt$

$\uparrow$
vseeno $\epsilon=0$
vseeno na $\mathbb{R} \times [0, \pi]$

$$0 = \int_{\partial D} f(z) dz = \int_{k_1} + \dots + \int_{k_4} \xrightarrow{\epsilon \rightarrow 0} 2iI + 0 - i\pi$$

$$\Rightarrow I = \frac{i\pi}{2i} = \frac{\pi}{2}$$

$$a > 0 \quad p \in (0, 1) \quad 0 < \epsilon \in \mathbb{R}$$

S pomočjo integracije po 1D izračunaj

$$I_1 = \int_0^{\infty} x^{p-1} \cos(ax) dx$$

$$I_2 = \int_0^{\infty} x^{p-1} \sin(bx) dx$$

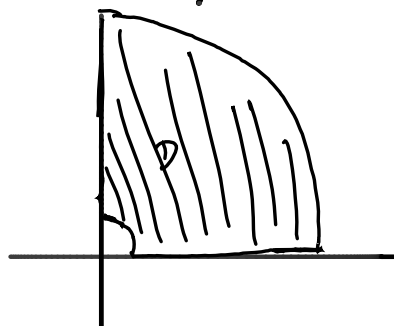
$$D = \{z; |z| \in (\epsilon, R) \wedge \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$$

$$f(z) = z^{p-1} e^{iaz}$$

holomorfnost:

$$e^{iaz} \in \mathcal{O}(C)$$

$$p = \frac{1}{2}: z^{p-1} = z^{-\frac{1}{2}} = \frac{1}{\sqrt{z}}$$



$$\sqrt{-1} = \pm i$$

$$\alpha \in \mathbb{C}, z \neq 0$$

$$z^\alpha = e^{\alpha \ln z}$$

Definirujmo kompleksne potence

$\ln z$ je nedoločen do konstante $2k\pi i$

$$z^\alpha = e^{\alpha (\ln_0(z) + 2k\pi i)} = (z^\alpha)_0 e^{2\alpha k\pi i}$$

če $\alpha \in \mathbb{Z}$ je z^α enolično definirana

z^α je holomorfno definirana, ko se odločimo za vejo $\ln(z)$

$$\uparrow \arg z \in (\varphi_0, \varphi_0 + 2\pi)$$

$\mathbb{C}$ preceznovomeri φ_0

Nas prihrani: ni vključeno, kar ne
bi bilo
zvezo v $\arg = \pi$

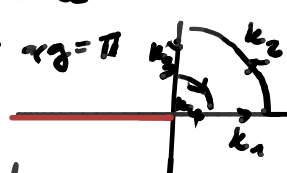
$$\arg z \in (-\pi, \pi)$$

Produkt holomorfni je
holomorfa

z^{p-1} holomorfa

$$\mathbb{C} - (-\infty, 0]$$

Izberemo si rez v smeri π ,
da ne prečka našega
domena



$$\int_{k_1}^R f(z) dz = \int_0^R x^{p-1} e^{iax} dx = \int_0^R x^{p-1} (\cos ax + i \sin ax) dx =$$

$$\xrightarrow[\epsilon \rightarrow 0]{R \rightarrow \infty} I_1 + i I_2$$

$$k_3: z = ix \quad dz = i dx \quad \ln(ix) = \ln y + i \frac{\pi}{2}$$

$$z^{p-1} = e^{(p-1) \ln(z)} = e^{(p-1)(\ln x + i \frac{\pi}{2})} =$$

$$= e^{(p-1) \ln x} e^{(p-1) i \frac{\pi}{2}} = x^{p-1} \cdot e^{(p-1) i \frac{\pi}{2}}$$

$$\int_{R_0}^R x^{p-1} e^{(p-1) i \frac{\pi}{2}} \cdot e^{iaix} = \int_{R_0}^R x^{p-1} e^{i(p-1) \frac{\pi}{2}} e^{iaix} dx =$$

$$= - e^{i(p-1) \frac{\pi}{2}} \int_0^R x^{p-1} e^{-ax} dx \xrightarrow[\epsilon \rightarrow 0]{R \rightarrow \infty}$$

$$= - e^{i(p-1) \frac{\pi}{2}} \int_0^{\infty} x^{p-1} e^{-ax} dx = \dots \int_0^{\infty} \frac{1}{a^{p+1}} e^{-t} dt$$

$$t = ax$$

$$dt = a dx$$

$$= - \frac{e^{i(p-1) \frac{\pi}{2}}}{a^p} \Gamma(p)$$

$$\int_{k_2} z^{p-1} e^{iaz}$$

$$z^{p-1} = (Re^{i\varphi})^{p-1} =$$

$$e^{(p-1) \ln(Re^{i\varphi})}$$

$$dz = i R e^{i\varphi} d\varphi$$

$$\ln(Re^{i\varphi}) = \ln R + i\varphi$$

$$= e^{\ln R (p-1) + (p-1) i\varphi} = R^{p-1} \cdot e^{i(p-1)\varphi}$$

$$\int_{k_2} \dots \int_0^{\frac{\pi}{2}} R^{p-1} e^{i(p-1)\varphi} e^{ia R (\cos \varphi + i \sin \varphi)} i R e^{i\varphi} d\varphi$$

$$|S_2| \leq \int_0^{\frac{\pi}{2}} | \dots | d\varphi = \dots \text{dos vrednosti} = 1$$

$$R^p \int_0^{\frac{\pi}{2}} e^{-a R \sin \varphi} d\varphi \leq \int_0^{\frac{\pi}{2}} R^p e^{-\frac{2\varphi a R}{\pi}} d\varphi =$$

$$\sin \varphi \geq \frac{2\varphi}{\pi}$$

$$= - R^p \frac{\pi}{2aR} (e^{-aR} - 1) \rightarrow 0$$

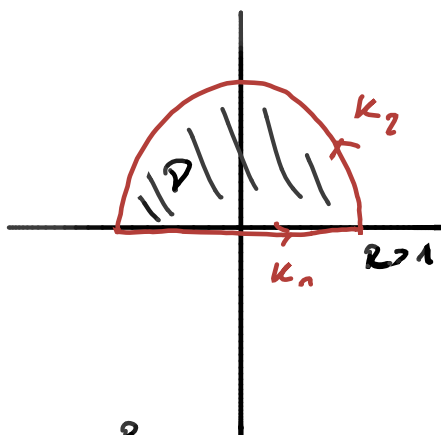
$$0 < p < 1$$

8.5

Tomada
one nalgae

$$I = \int_0^{\infty} \frac{\cos x}{1+x^2} dx$$

$$f(z) = \frac{e^{iz}}{1+z^2}$$



$$\int_{K_1} f(z) dz = \int_{-R}^R \frac{e^{ix}}{1+x^2} dx = \int_{-R}^R \frac{\cos x + i \sin x}{1+x^2} dx =$$

$$= \int_{-R}^R \frac{\cos x}{1+x^2} dx + i \int_{-R}^R \frac{\sin x}{1+x^2} dx = 2 \int_0^R \frac{\cos x}{1+x^2} dx \xrightarrow{R \rightarrow \infty} 2I$$

//
O.K. like

$$\int_{K_2} f(z) dz = \int_0^{\pi} \frac{e^{i(r e^{i\varphi} + \varphi)}}{1+r^2 e^{2i\varphi}} i r e^{i\varphi} d\varphi =$$

$$\left| i \int_0^{\pi} \frac{e^{-rs \varphi} \cdot e^{i r \cos \varphi}}{1+r^2 e^{2i\varphi}} e^{i\varphi} d\varphi \right| \leq$$

$$r \int_0^{\pi} 1 \dots d\varphi$$

$$|1+r^2 e^{2i\varphi}| \geq |1-|r^2 e^{2i\varphi}|| = |1-r^2| \geq r^2-1$$

$r > 1$
↓

$$\leq r \int_0^{\pi} e^{-rs \varphi} \frac{1}{r^2-1} d\varphi = \frac{r}{r^2-1} \int_0^{\pi} e^{-rs \varphi} d\varphi \leq \frac{r}{r^2-1} \xrightarrow{r \rightarrow \infty} 0$$

f ima singularnost v i ∈ D

$$f(z) = \frac{e^{iz}}{(z-i)(z+i)} = \frac{\frac{e^z}{z+i}}{z-i}$$

$$\boxed{\int_{\partial D} \frac{g(z)}{z-a} dz = 2\pi i g(a)}$$

$$\int_{\partial D} \frac{e^z}{z+i} dz = 2\pi i \frac{e^{-1}}{2i} = \frac{\pi}{e}$$

$$\frac{\pi}{e} = \int_{\partial D} f(z) dz = \int_{K_1} + \int_{K_2} \xrightarrow{R \rightarrow \infty} 2I + 0 \Rightarrow I = \frac{\pi}{2e}$$

Ali obstaja funkcija f holomorfna v
 delici 0, da za velike neravne števila $n \in \mathbb{N}$
 velja:

a) $f\left(\frac{1}{n}\right) = \sin\left(\frac{n\pi}{2}\right)$

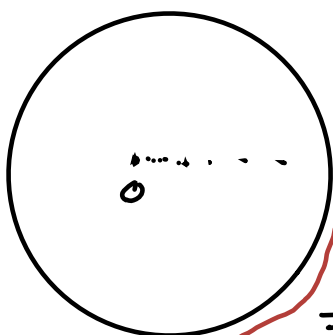
b) $f\left(\frac{1}{n}\right) = \frac{1}{2n+1}$

c) $n^{-\frac{5}{2}} < |f\left(\frac{1}{n}\right)| < 3n^{-\frac{5}{2}}$

g) $\sin \frac{n\pi}{2} = \begin{cases} 0 & n \text{ sodo} \\ 1 & 4k+1 \\ -1 & 4k+3 \end{cases}$

V vsaki delici 0 ne f vrednosti 0 ali 1
 $\Rightarrow f$ ni zvezna $\Rightarrow f$ ni holomorfna

b)



$A = \{\frac{1}{n}, n \in \mathbb{N}\}$ je množica
 s številiscem v D

$\Rightarrow$ če $\exists f$ je natanko
 delovanje
 princip identičnosti

$f\left(\frac{1}{n}\right) = \frac{1}{2n+1} \dots \rightarrow f(z) = \frac{1}{\frac{z}{2} + 1} = \frac{2}{2+z}$

holomorfna v
 delici 0

c) če f obstaja ima razvoj v
 Taylorjevo vrsto v delici 0

$f(z) = c_0 + c_1 z + c_2 z^2 + \dots$

$n^{-\frac{5}{2}} < |c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \dots| < 3n^{-\frac{5}{2}}$

$\downarrow$ $\downarrow$ $\downarrow$
 $c \leq |c_0| \leq 0$

$n^{-\frac{3}{2}} < |c_1 + \frac{c_2}{n} + \dots| < 3n^{-\frac{3}{2}}$ $\cdot n$

$\Rightarrow c_1 = 0$

$\Rightarrow c_2 = 0$

$n^{\frac{1}{2}} \leq |c_3 + \frac{c_4}{n} + \dots| \leq n^{\frac{1}{2}}$ $\cdot n$

$\downarrow$ $\downarrow n \rightarrow \infty$ $\downarrow$
 ∞ c_3 ∞

*

Kaj pa $2z^{\frac{5}{2}}$?

je ne vselem prerezu, ampak ne
 moremo pa je posplošiti na celo
 delico 0

Razvij $f(z) = \sin z$ duži tačke i

$$f(z) = c_0 + c_1(z-i) + c_2(z-i)^2 + \dots$$

$$c_k = \frac{f^{(k)}(i)}{k!} \quad \text{u zploten}$$

$$w = z - i$$

$$\sin(z) = \sin(w+i) = \sin(w)\cos(i) + \sin(i)\cos(w)$$

$$= \cos i \cdot \sum_{k=0}^{\infty} \frac{w^{2k+1}}{(2k+1)!} (-1)^k + \sin i \cdot \sum_{k=0}^{\infty} \frac{(-1)^k w^{2k}}{(2k)!}$$

$$= \cos i \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (z-i)^{2k+1} + \sin i \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (z-i)^{2k}$$

$$\sin(x+iy) = \sin x \cos yi + \sin yi \cos x$$

$$\sin iy = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (iy)^{2k+1} =$$

$$= \sum_{k=0}^{\infty} i \frac{y^{2k+1}}{(2k+1)!} (-1)^k = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot i \cdot y^{2k+1}}{(2k+1)!} =$$
$$= i \sum_{k=0}^{\infty} \frac{y^{2k+1}}{(2k+1)!} = i \operatorname{sh}(y)$$

$$\cos(iy) = \sum_{k=0}^{\infty} \frac{(-1)^k (iy)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{y^{2k}}{(2k)!} = \operatorname{ch}(y)$$

$$\sin(x+iy) = \sin x \operatorname{ch} y + i \operatorname{sh} y \cos x$$

$$\sin i = 0 + i \operatorname{sh} 1 = i \operatorname{sh} 1$$

$$y=1$$

$$x=0$$

$$\cos(x+iy) = \cos x \cos yi - \sin x \sin yi =$$
$$\cos x \operatorname{ch} y - i \sin x \operatorname{sh} y$$

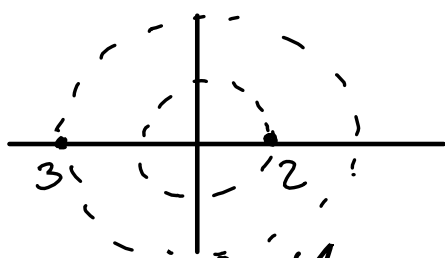
$$\cos i = \operatorname{ch} 1$$

$$x=0$$

$$y=1$$

Laurentova vrsta

$$f \in \mathcal{O}(D) \Rightarrow f(z) = \underbrace{\sum_{n=-\infty}^{-1} a_n (z-a)^n}_{\text{glavni del}} + \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{regularni del}}$$



$$f(z) = \frac{z+1}{z^2+z-6} = \frac{A}{(z+3)} + \frac{B}{(z-2)}$$

$$|z| < 1 \Rightarrow B=1 \quad A=1$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad |z/3| < 1 \Rightarrow |z| < 3$$

$$\frac{1}{3+z} = \frac{1}{3} \frac{1}{1 - (-z/3)} = \frac{1}{3} \sum_{n=0}^{\infty} \left(-\frac{z}{3}\right)^n$$

$$|z| < 3 \Rightarrow \frac{1}{z+3} = \frac{1}{3} \sum_{n=0}^{\infty} \left(-\frac{z}{3}\right)^n$$

$$|z| < 2 \Rightarrow -\frac{1}{z} \frac{1}{1 - z/2} = -\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

$$\begin{aligned} z \in A(0; 0, 2) &\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} z^n - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \\ &= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{3^{n+1}} - \frac{1}{2^{n+1}} \right) z^n \end{aligned}$$

$$z \in A(0; 2, 3)$$

$$\frac{1}{z-2} = \frac{1}{z} \frac{1}{1 - \frac{2}{z}} \quad \text{razvoj nazven}$$

$$g = \frac{2}{z} \quad |2/z| < 1 \quad 2 < |z|$$

$$= \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n = \sum_{n=0}^{\infty} \frac{2^{n+1}}{z^{n+1}}$$

$$m = -(n+1) \Rightarrow n = -m-1$$

$$z \in A(0; 2, 3) \Rightarrow f = \sum_{n=-\infty}^{-1} \frac{1}{2^{n+1}} z^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} z^n$$

$$z \in A(0; 3, \infty)$$

$$\frac{1}{z+3} = \frac{1}{z} \frac{1}{1 + \frac{3}{z}} \quad g = \left(-\frac{3}{z}\right)$$

$$\frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{z^n} = \quad |z| > 3$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{z^{n+1}}$$

$$f(z) = \sum_{n=-\infty}^{-1} \left((-3)^{-n+1} + 2^{-(n+1)} \right) z^n$$

Pogledaj:

$$f(z) = \frac{p(z)}{q(z)} \quad \text{p.g. polinom}$$

→ razcepimo ulomke

$$\frac{1}{(z-a)^k} = (z-a)^{-k} =$$

↑
binomska vrsta
 $(1+g)^{\alpha}$

2. Določiti glavni del LV

$$f(z) = \frac{\cos z}{z - \sin z} \quad ; a = 0$$

$$f(z) = \frac{1}{(z+1)(e^{i\pi z} + 1)} \quad ; a = -1$$

izolirani
singularnosti
v a

$$f \in O(D - 2a^2)$$

$\hookrightarrow$ izolirane singularnosti

1) Bistvena singularnost

glavni del ima neskončno
nenišelnih členov

2) Pol stopnje $N \in \mathbb{N}$

glavni del je do N

3) a je odpravljiva singularnost

$\Rightarrow$ glavni del $= 0$

$$f(z) = \frac{1 - \frac{z^2}{2} + \frac{z^4}{4!} - \dots}{z - \left(\cancel{z} - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right)} =$$

$$= \frac{1 - \frac{z^2}{2} + \frac{z^4}{4!} + \dots}{\frac{z^2}{3!} - \frac{z^5}{5!} + \dots} = \frac{1}{z^3} \left(\overbrace{1 - \frac{z^2}{2} + \dots}^{g(z)} \cdot \frac{1 - \frac{z^2}{2} + \dots}{\frac{1}{3!} - \frac{z^2}{5!} + \dots} \right)$$

$g(z)$ holomorfen u. d. d. 0. n. i. m. a. n. i. d. v. 0.

$\Rightarrow f(z)$ ima pol 3-stupnje

$$GD: \frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z}$$

$$f(z) = \frac{1 - \frac{z^2}{2} + \dots}{z^3 \left(\frac{1}{3!} - \frac{z^2}{5!} + \dots \right)} = \frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \underbrace{\dots}_{\text{reg. del}}$$

$$1 - \frac{z^2}{2} + \dots = (a_{-3} + a_{-2}z + a_{-1}z^2 + \dots) \left(\frac{1}{3!} - \frac{z^2}{5!} + \dots \right)$$

$$z^0: \frac{a_{-3}}{3!} = 1 \quad z^2: \frac{a_{-3}}{5!} + \frac{a_{-1}}{3!} = -\frac{1}{2}$$

$$z: \frac{a_{-2}}{3!} = 0$$

...

$$a_{-1} = \frac{-9 \cdot 6}{20}$$

$$a_{-3} = 3! = 6$$

$$a_{-2} = 0$$

$$a_{-1} = \frac{1}{2}$$

$$GD: \frac{6}{z^3} - \frac{9 \cdot 6}{20z} = \frac{27}{10z}$$

$$b) f(z) = \frac{1}{(z+1)} (e^{i\pi z} + 1)$$

$$e^{i\pi z} \text{ razvijemo oko } a = -1$$

$$w = z - a = z + 1$$

$$e^{i\pi(w-1)} = e^{i\pi w} e^{-i\pi}$$

$$(-1) \sum_{n=0}^{\infty} \frac{(i\pi w)^n}{n!} = -1 - \pi w - \dots$$

$$e^{i\pi z} + 1 = -\pi w - \dots = -i\pi(z+1)$$

$$f(z) = \frac{1}{(z+1) (-i\pi(z+1) - \dots)} =$$

$$\frac{1}{(z+1)^2 (-i\pi - \dots)} = \frac{a_{-2}}{(z+1)^2} + \frac{a_{-1}}{z+1} + \dots$$

od druge stranice

$$1 = a_{-2}(-i\pi - \dots) + a_{-1}(z+1) \left(\frac{i\pi^2}{2}(z+1) \right)$$

$$(z+1)^0: a_{-2} i\pi$$

$$(z+1)^1: a_{-2} \frac{i\pi^2}{2} + a_{-1} i\pi \quad ??$$

$$GD: \frac{1}{\pi(z+1)^2} + \frac{1}{2(z+1)}$$

$$\text{Res}(f, -1) = a_{-1} = \frac{1}{2}$$

$$f(z) = \frac{z}{z^3 - 3z + 2}$$

Rozwij okolic
a = -1

izracunaj

$$\int f(z) dz$$

$$|z| = 4$$

$$z^3 - 3z + 2 =$$

$$w = z - 1 \quad z = w + 1$$

$$= w^3 + 3w^2 + 3w + 1 - 3w - 3 + 2 =$$

$$= w^3 + 3w^2 - 2 = w(w^2 + 3w - 2)$$

$$= w \underline{\quad \quad \quad}$$

$$= w^2(w+3)$$

$$w+1; w+3=1 \quad \frac{-2}{w+3}$$

$$f(z) = \frac{w+1}{w^2(w+3)} = \frac{1}{w^2} \frac{w+1}{w+3} =$$

$$\frac{1}{w^2} \left(1 - \frac{2}{w+3} \right)$$

$$1 - \frac{2}{w+3} = 1 - \frac{2}{\frac{w}{3} + 1} = -\frac{2}{3} \sum_{n=0}^{\infty} (-1)^n \frac{w^n}{3^n}$$

...

Smun

Formula:

$\int_{\partial D} f(z) dz$... f holomorphic na
 okoli ∂D
 razen v končnem številu
 točk v notranjosti D

$$\int_{\partial D} f(z) dz = \sum_{a \in D} \text{Res}(f, a)$$

$\uparrow$
 singularnost

f pol stopnje N v $a \Rightarrow$

$$\text{Res}(f, a) = \frac{1}{(N-1)!} \lim_{z \rightarrow a} (f(z) (z-a)^N)^{(N-1)}$$

$$\int \frac{dz}{z^2 + 9}$$

$$|z - z_i| = 2$$

$$f(z) = \frac{1}{(z + 3i)(z - 3i)}$$

$$a_1 = 3i$$

$$a_2 = -3i$$

singular point
pole of order 1

$$I = 2\pi i \operatorname{Res}(f, a_1)$$

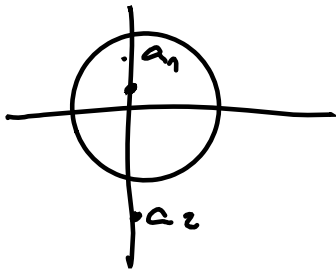
$$\operatorname{Res}(f, 3i) = \lim_{z \rightarrow 3i} \frac{1}{z + 3i} = \frac{1}{6i}$$

$\uparrow$
 $f(z)(z - 3i)$

$$\left(\operatorname{Res}(f, -3i) = -\frac{1}{6i} \right) \quad I = \frac{\pi}{3}$$

$$b) \int_{|z - \frac{i}{2}| = 1} \frac{dz}{(z^2 + 1)^3}$$

$$f(z) = \frac{1}{(z - i)^3 (z + i)^3} \quad \begin{matrix} a_1 = i \\ a_2 = -i \end{matrix}$$



$$I = 2\pi i \operatorname{Res}(f, a_1)$$

$$\operatorname{Res}(f, a_1) =$$

$$N = 3$$

$$\operatorname{Res}(f, a_1) = \frac{1}{2!} \lim_{z \rightarrow i} \left(\frac{1}{(z + i)^3} \right)'' =$$

$$\left((z + i)^{-3} \right)' = -3 \left((z + i)^{-4} \right)' = 12 (z + i)^{-5}$$

$$\operatorname{Res}(f, a_1) = 6 \cdot (2i)^{-5} = \frac{6}{2^5 i} = \frac{3}{2^4 i} = \frac{3}{16i}$$

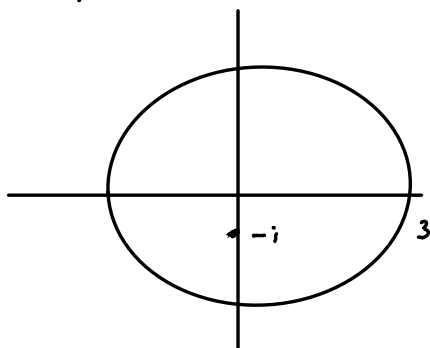
$$I = \frac{\pi}{8}$$

$$c) \int \sin\left(\frac{z}{z+i}\right) dz$$

$$|z|=3$$

$$f(z) = \sin\left(\frac{z}{z+i}\right)$$

$$a = \frac{-i}{1+i}$$



$$w = z+i$$

$$z = w-i$$

$$f(z) = \sin\left(\frac{w-i}{w}\right) = \sin\left(1 - \frac{i}{w}\right) =$$

$$= \sin 1 \cos\left(\frac{i}{w}\right) - \sin\left(\frac{i}{w}\right) \cos(1) =$$

$$= \sin 1 \left(1 - \frac{\left(\frac{1}{w}\right)^2}{2} + \dots\right) - \cos(1) \left(\frac{i}{w} - \frac{\left(\frac{i}{w}\right)^3}{6} + \dots\right)$$

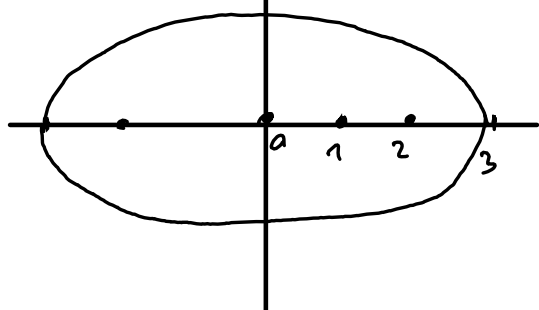
$$\text{Res}(f, i) = C_{-1} \propto_{i=i} w^{-1}$$

$$\Rightarrow \text{Res}(f, i) = -\cos(1) i$$

$$\int \frac{1}{z} e^{\frac{1}{1-z}} dz$$

$$|z-2| + |z+2| = 6$$

"
 $z=$



$$a_1 = 0$$

$$a_2 = 1$$

↖

$$I = (\text{Res}(f, 1) + \text{Res}(f, 0)) 2\pi i$$

$$\text{Res}(f, 0) = \lim_{z \rightarrow 0} (f(z) \cdot z) = \lim_{z \rightarrow 0} e^{\frac{1}{1-z}} = e$$

$$\text{Res}(f, 1) = ? \text{ biogene singularit\u00e4t}$$

$$z \neq 1 \vee z = 1$$

$$\frac{1}{z} \text{ v \u00f6keli: } 1$$

$$\frac{1}{z} = \frac{1}{z-1+1} = \sum_{n=0}^{\infty} (z-1)^n (-1)^n \quad |z-1| < 1$$

$$e^{\frac{1}{1-z}} = \sum_{n=0}^{\infty} \frac{(\frac{1}{1-z})^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{(z-1)^n}$$

$$\frac{1}{z} \cdot e^{\frac{1}{1-z}} = (1 - (z-1) + (z-1)^2 - \dots) \left(1 - \frac{1}{z-1} + \frac{1}{2(z-1)^2} - \frac{1}{3!(z-1)^3} + \dots \right)$$

$$= \dots + \frac{1}{z-1} \left(-1 - \frac{1}{2} - \frac{1}{3!} - \frac{1}{4!} - \dots \right) + \dots$$

$$e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$$

$$= -(e-1) = 1-e$$

$$I = 2\pi i (e + 1 - e) = 2\pi i$$

$$z = e^{ix}$$

$$dx = \frac{dz}{iz}$$

$$\cos x = \frac{z + z^{-1}}{2} = \frac{z^2 + 1}{2z}$$

$$\sin x = \frac{z^2 - 1}{2iz}$$

$$f(z) = \frac{h(z)}{g(z)}$$

$$\operatorname{Res}(f, z_0) = \frac{h(z_0)}{g'(z_0)}$$

Št ničel z $p(z) = z^5 + 7z^4 - z^2 + 2$

Na a) $\Delta(0,1)$ b) $\Delta(\overline{0,1})$ c) $\Delta(0,2,10)$

Rouchierjev izrek:

$D \subset \mathbb{C}$ omejeno lepo območje (gledat ∂D)

$f, F \in \mathcal{O}(\text{okolica } D)$

$\forall z \in \partial D: |f(z)| < |F(z)|$
(na ∂D)

$\Rightarrow$ Št ničel vednoostah z F je enako številu ničel z $f+F$

a) $p(z) = z^5 + 7z^5 - z^2 + 2 = f(z) + F(z)$

$z \in \partial D \Leftrightarrow |z| = 1$

$F(z) = 7z^5 \Rightarrow |F(z)| = 7$

$f(z) = z^5 - z^2 + 2 \Rightarrow |f(z)| \leq 1 + 1 + 2 = 4$

R.I. $\Rightarrow$ Št ničel $p(z)$ na $\Delta(0,1)$ je 4

b)

Opomba

$z \in \partial D: |F(z)| > |f(z)|$

$|F(z) + f(z)| \geq |F(z)| - |f(z)| > 0$

$\Rightarrow F(z) + f(z) \neq 0$ za $z \in \partial D$

$\Rightarrow \underbrace{g(z)}$

$g(z)$ nima ničel na ∂D

c) $\Delta(0,2,10)$

$p(z) = z^5 + 7z^4 - z^2 + 2$

$z \in \partial D \Leftrightarrow |z| = 2 \vee |z| = 10$

$D = \underbrace{\Delta(0,10)}_{D_1} - \underbrace{\Delta(0,2)}_{D_2}$

preštujemo nile na D_1 in potem na D_2

$D_1: |z| = 10$

$F = z^5 \Rightarrow |F(z)| = 10^5$

$f(z) = 7z^4 - z^2 + 2 \Rightarrow |f(z)| \leq 7 \cdot 12$

~~$D_2: |z| = 2$~~

$F(z) = z^4$

$F(z) = 7 \cdot 2^4$

$|f(z)| \leq 2^5 + 4 + 2$

$\Rightarrow$ število:

$5 - 4 = 1$

Dokazi da ima $z \cdot \sin z = 1$ same realne
resitve

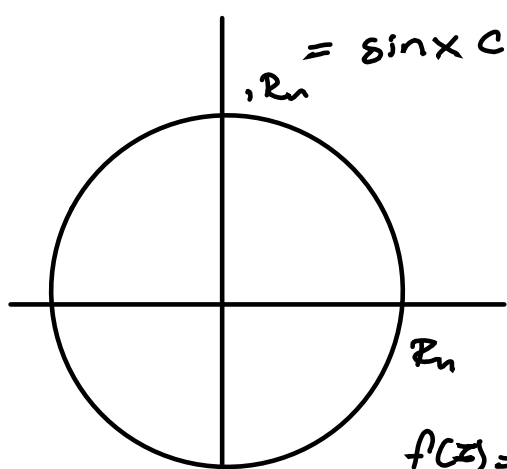
$\sin z$ ima same realne nicle

$$\sin z = 0$$

$$z = \pi k, k \in \mathbb{Z}$$

$$\sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy) =$$

$$= \sin x \operatorname{ch} y + i \operatorname{sh}(y) \cos(x)$$



$$D = \Delta(0, R_n)$$

$$z_n = 2n\pi + \frac{\pi}{2}$$

$$f(z) = z \sin z - 1$$

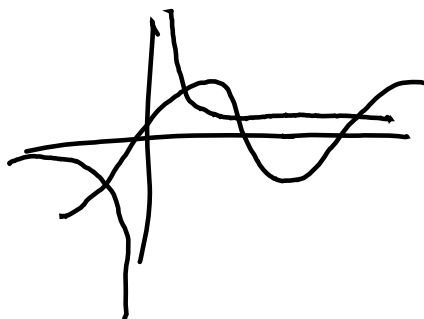
Re nicle z na D

= št vsen nicle

Realne nicle $z=x$

$$x \cdot \sin x = 1 \quad x \neq 0$$

$$\sin x = \frac{1}{x}$$



6 re nicle na $(0, z_n)$:

$$z_n + 1$$

Re nicle na D :

$$z \cdot \sin z - 1 = 0$$

$$f(z) = z \cdot \sin(z)$$

$$|g(z)| = -1$$

$$z \in D \Leftrightarrow |z| = R_n$$

$$|z \cdot \sin(z)| \leq R_n \cdot \frac{1}{1} \cdot \frac{1}{1}$$

$$g(z) = 1$$

Najmanjša vrednost $| \sin(x+iy) |$

$$u(x,y) = | \sin(x+iy) |^2 =$$

$$\sin^2 x \operatorname{ch}^2 y + \operatorname{ch}^2 y \cos^2 x$$

$$\text{minimum na } \Delta(0, R_n) \quad x^2 + y^2 = R_n^2$$

Verzani ekvanci:

$$L = \sin^2 x \operatorname{ch}^2 y + \cos^2 x \operatorname{sh}^2 y - \alpha (x^2 + y^2 - R_n^2)$$

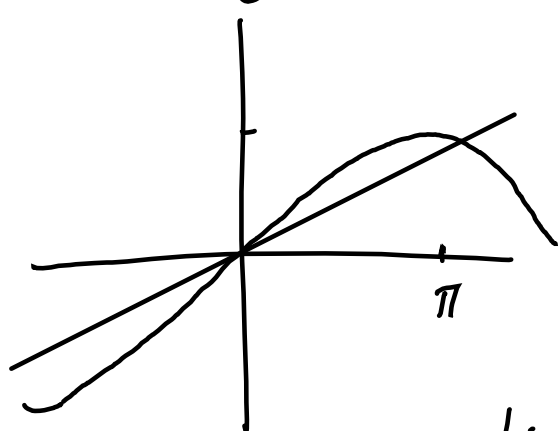
$$L_x = 2 \sin x \cos x \operatorname{ch}^2 y - \operatorname{sh}^2 y 2 \sin x \cos x \dots$$

$$\Rightarrow \sin 2x = 2\alpha x$$

$$\operatorname{sh} 2y = 2\alpha y$$

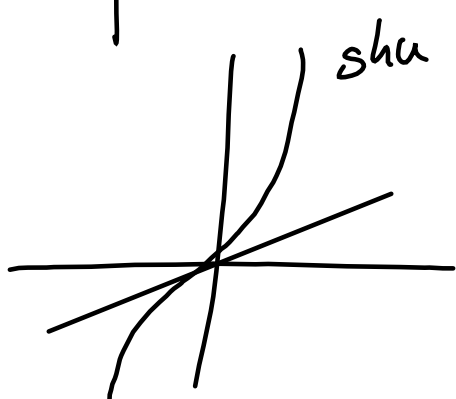
$$2x = t \\ \sin t = \alpha t$$

$$2y = u \\ \operatorname{sh} u = \alpha u$$



$\alpha > 1 \Rightarrow$ sekla
sežno v izhodiscu

$\alpha < 1 \Rightarrow$



$\alpha > 1 \Rightarrow$ sekla
nadje

$\alpha < 1 \Rightarrow$ sekla nadje
sežno v izhodiscu

v rektum
pamau

je suu ene
izhodiscu

$$\alpha > 1 \Rightarrow x=0 \Rightarrow T(0, \pm R_n)$$

$$\alpha < 1 \Rightarrow y=0 \Rightarrow (\pm R_n, 0)$$

$$\alpha = 1 \Rightarrow (x=x=0) \Rightarrow *$$

$$F(z) = z + \sin z \quad f(z) = -1$$

$$|F(z)| > f(z)$$

1 nudi st 02

4n ničel stopnje 1

4n+2 ničel

← sajeda > številom
realnih rešitev
saj so vse rešitve

Konformne preslikave

ohranja kote in orientacijo
↖ med krivuljami

= holomorfne z neničelnim odvodom

$$f: D_1 \longrightarrow D_2 \quad D_1, D_2 \subseteq \mathbb{C}$$



biholomorfna ekvivalenca

f bijektivna, holomorfna, f^{-1} holomorfna

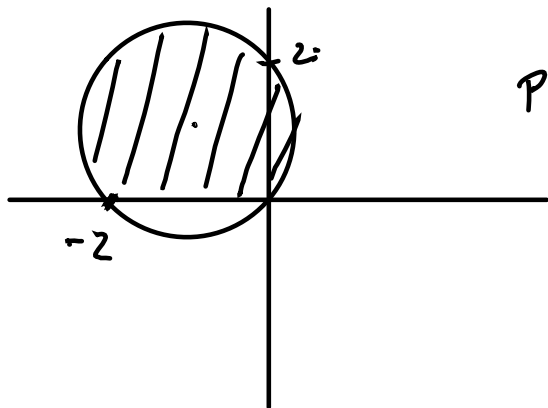
$D \subseteq \mathbb{C}$ je biholomorfno ekvivalenten Δ



$D \neq \mathbb{C}$ $\wedge$ D enostavna povezana
(nima lukenj)

Uagotovi ali je D bih. ekv Δ , če je,
deloči se biholomorfna

$$a) D = \{z; |z - i + 1| < \sqrt{2}, \operatorname{Im} z > 0\}$$



← Rimanov
upodobitveni:
Po RUI obstaja izrek
 $f: D \rightarrow D$

$$g(z) = \frac{az + b}{cz + d}$$

lomljena linearna
preslikava

$\{\text{krožnice, premice}\} \longrightarrow \{\text{krožnice, premice}\}$

$$\mathbb{CP}^1 = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

Premice so krožnice
skozi ∞

$$g: \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}} \text{ biholomorfna}$$

Poišči LUP

$$g: g(-2) = 0 \quad g(-1) = 1 \quad g(0) = \infty$$

$$\frac{az+b}{cz+d}$$

$$\frac{a(z+2)}{cz}$$

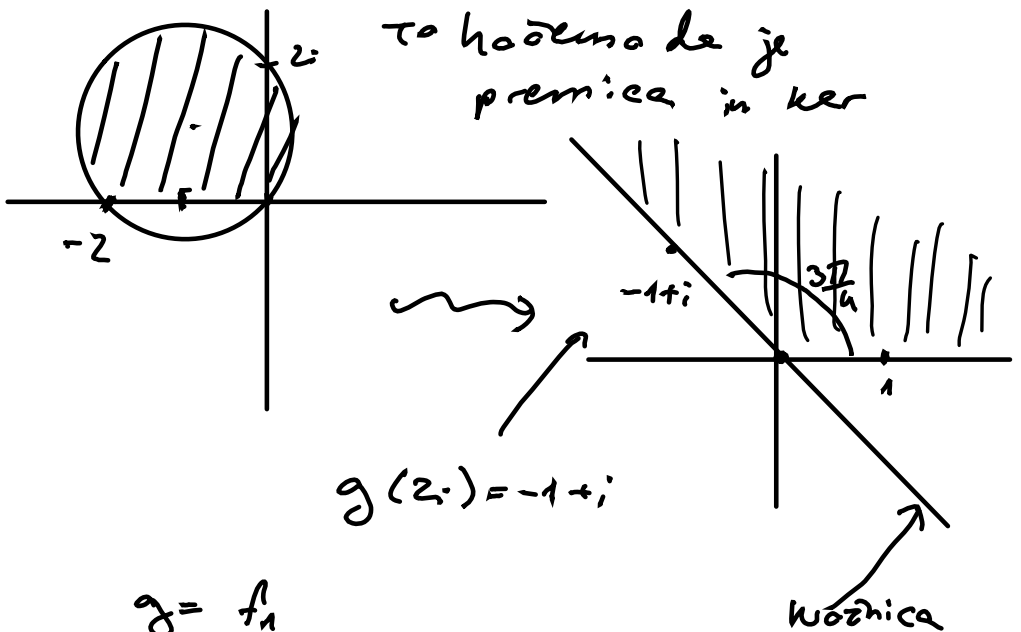
$$a=1$$

$$\frac{-1+2}{c(-1)} = 1$$

$$1 = -c$$

$$g = \frac{z+2}{-z}$$

To smo gledali ker



$$g = f_1$$

$$g(D) = D_1$$

Naprej

$$f_2: D_1 \rightarrow H$$

$$\frac{3\pi}{4} \dots \pi$$

$$y \dots x$$

$$x = \frac{4\pi}{3\pi} = \frac{4}{3}\rho$$

$$z \mapsto |z| e^{\frac{4}{3}\arg(z)}$$

$$\text{lahko tudi } z \mapsto z^{\frac{4}{3}}$$

$$z^\alpha = e^{\alpha \ln z}$$

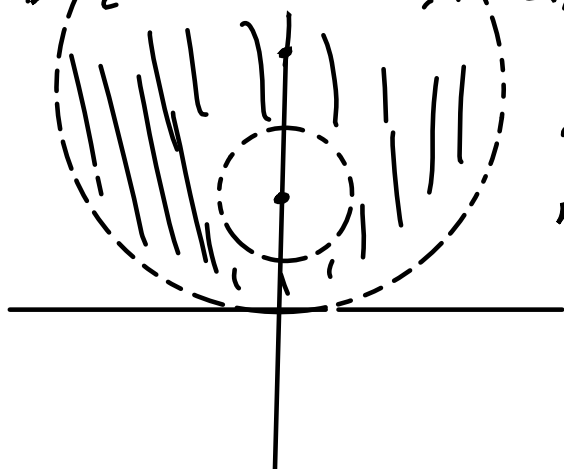
$$f_0(z): H \rightarrow \triangle$$

$$f(z) = \frac{iz+1}{iz-1}$$

$$f = f_0 \circ f_2 \circ f_1$$

b)

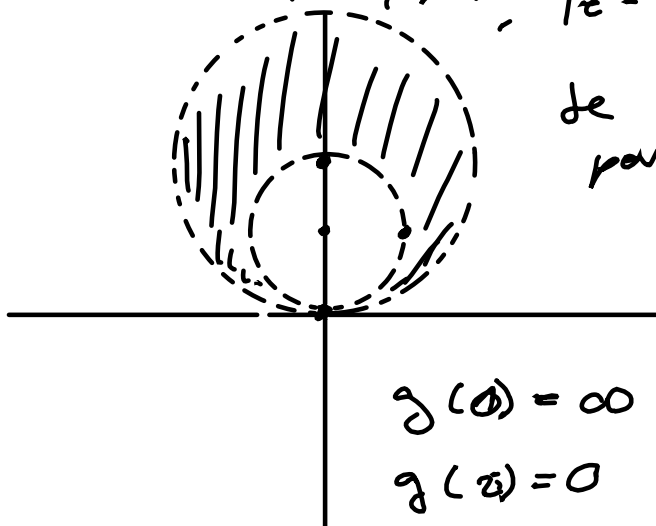
$$D = \{z; |z-i| > \frac{1}{2}, |z-2i| < 2\}$$



D ni enostavno
povezano $\Rightarrow$
ni bih oku $\triangleleft$

c)

$$D = \{z; |z-i| > 1; |z-2i| < 2\}$$



je enostavno
povezano drugje

$$g(0) = \infty$$

$$g(i) = 0$$

$$g(i+1) = 1$$

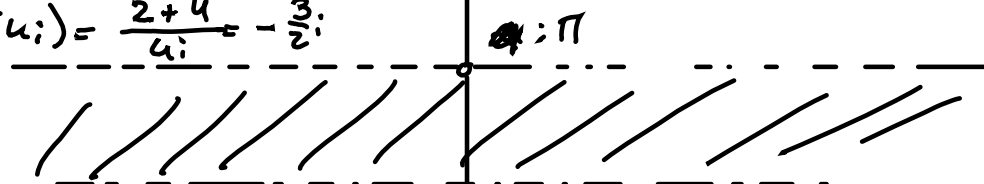
$$\frac{a(z-2i)}{cz}$$

$$\frac{a}{c} \frac{(1-i)}{1+i} = \frac{a}{c} \frac{1-2i-1}{2} = -\frac{ai}{c}$$

$$\frac{a}{c} = -i$$

$$g = \frac{2-z-i}{z}$$

$$g(4i) = \frac{2+4}{4i} = -\frac{3}{2}i$$



$$f_1: z \mapsto e^z$$

$$e^{x+iy} = e^x \cdot e^{iy} \quad \leftarrow y \text{ je kot}$$

$$y \in (0, \pi)$$

$$x \in (-\infty, \infty)$$

$$e^x \in (0, \infty)$$

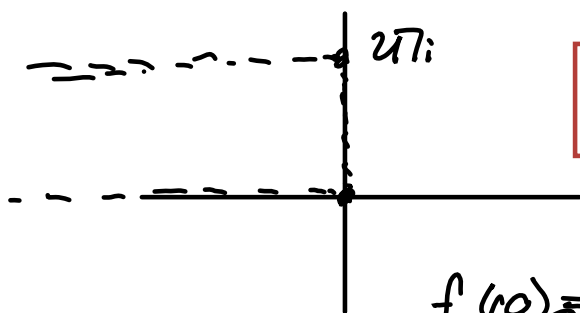
$$e^{iy} \in (0, 1)$$

$$s, k_2 \in H$$

$$f_0: D_2 \rightarrow \Delta$$

$$f = f_0 \circ f_3 \circ f_2 \circ f_1$$

$$D = \{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0 \text{ and } \operatorname{Im}(z) \in (0, 2\pi)\}$$



e^z rozbiega się

$$f_1: x \in (-\infty, 0) \quad f_1(0) = 0 \quad f_1(2\pi i) = \infty \quad : e^z$$

$$e^x : (0, 1)$$

$$e^{yi} : y \in (0, 2\pi)$$

$$f_2: z^{\frac{1}{2}}$$

$$(f_2 \circ f_1)(z) = e^{\frac{z}{2}}$$



$$f_3:$$

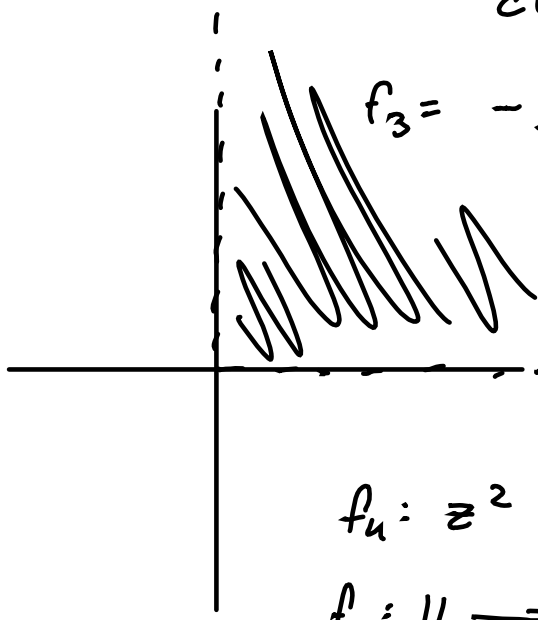
$$g(-1) = 0$$

$$g(1) = 1$$

$$g(1) = \infty$$

$$\frac{az+b}{cz+d}$$

$$\frac{a(z+1)}{c(z-1)} - \frac{a}{c} = 1$$

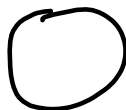


$$f_3 = -\frac{z+1}{z-1}$$

$$f_4: z^2$$

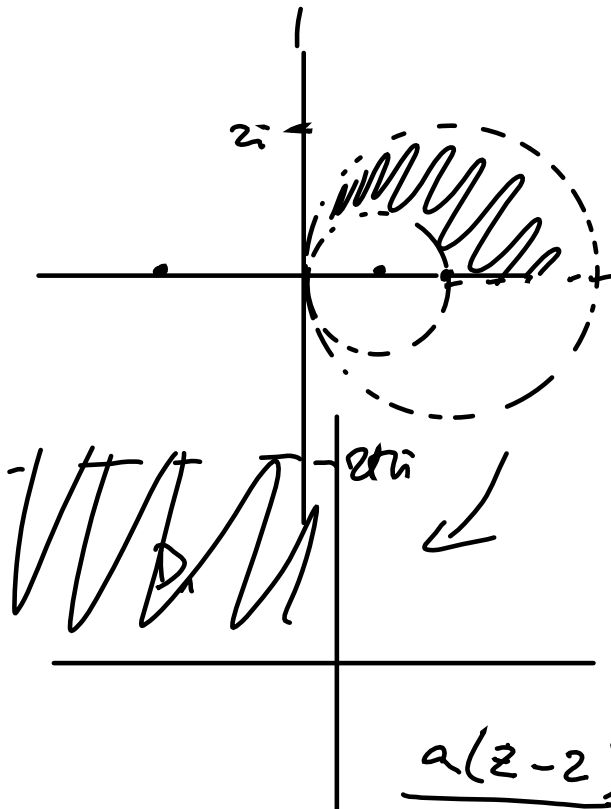
$$f_0: H \rightarrow \mathbb{A}$$

$$g = f_0 \circ f_4 \circ f_3 \circ f_2 \circ f_1$$



$$D = \{z \in \mathbb{C} \mid |z-1| > 1 \text{ and } |z-2| < 2 \text{ and}$$

$$\arg(z) \in (0, 2\pi)\}$$



$$f_1(0) = \infty$$

$$f_1(2) = 0$$

$$f_1(4) = 2\pi i$$

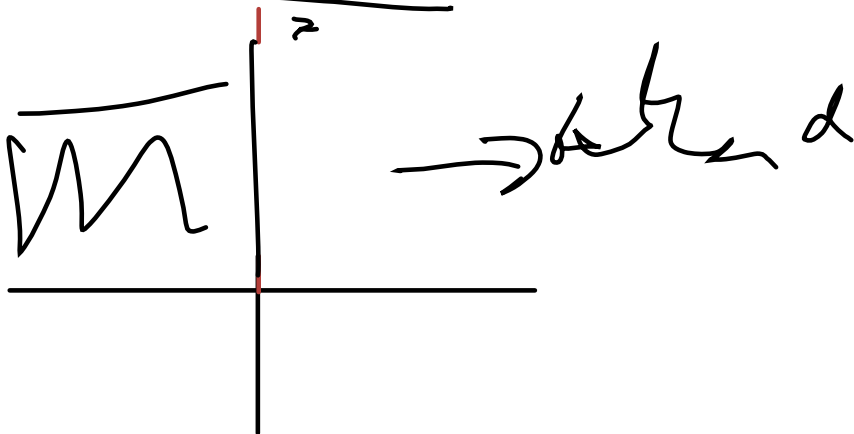
$$\frac{a(z-2)}{cz}$$

$$\frac{a}{c} \frac{4-2}{4} = \frac{a}{c} \frac{1}{2} = 2\pi i$$

$$\frac{a}{c} =$$

$$\frac{a}{c} = 4\pi i$$

$$\frac{4\pi i (z-2)}{z}$$



Določiti vse pare celih funkcij
za katere je $(f(z))^2 + (g(z))^2 = 1$

cela funkcija - holomorfná na $\mathbb{C}$
kandidata:

$$f(z) = \sin(h(z)) \quad g(z) = \cos(h(z))$$

$$(f(z))^2 + (g(z))^2 = \underbrace{(f(z) + ig(z))(f(z) - ig(z))}_{k(z)} = 1$$

k nima ničle

Izrek: Če je $k \in O(D)$ in je D enostavno
povezano, $\exists$ veja $\underbrace{\ln(k(z))}_{f(z)} \in O(D)$
 $(k(z) = e^{f(z)})$

$D = \mathbb{C}$ enostavno povezano
 $k(z) = e^{f(z)}$

$$2f(z) = e^{ih(z)} - e^{-ih(z)}$$

$$f(z) = c \cos(h(z))$$

$$g(z) = \sin(h(z))$$

f je holomorfná na povezaní odprti
okoli: $\infty \in \mathbb{CP}^1$

Določi f , če za $n \in \mathbb{N}$ $n \geq n_0$ velja

$$f(n) = \frac{n^2 n \sqrt{e}}{1+n^2}$$

f holomorfná v $\infty \Leftrightarrow f(\frac{1}{z})$ holomorfná v 0

Princip identitete