

ket: elemen² kelokan $\mathbb{Z}[i]$

6.10

asosiasi: $m+ni$

$$x = u(m+ni) = um + un i$$

ket: so konjugi:

$$(x+yi)(a+bi) = 1$$

$$xa - yb + i(ay + xb) = 1$$

$$ay + xb = 0 \quad xa - yb = 1$$

$$x = \frac{-ay}{b} \quad -\frac{a^2 y}{b} - yb = 1$$

$$y = \frac{-b}{a^2 + b^2} \quad -\frac{y}{b}(a^2 + b^2) = 1$$

$$x = \frac{a}{a^2 + b^2}$$

$$|a| \leq |a^2| \Rightarrow$$

$$a^2 + b^2 \in \{0, 1\} \Rightarrow a^2 + b^2 = 1$$

$$\Rightarrow a \in \{0, 1\}$$

$$b \in \{0, 1\}$$

$$\text{konjugi} \quad \{1, -1, i, -i\}$$

$$\text{asoc: } \{m+ni, -m-ni, n-mi, -n+mi\}$$

$$\mathbb{Z}[\sqrt{d}] = \{m+n\sqrt{d}; m, n \in \mathbb{Z}\}$$

1) Pokaži $\mathbb{Z}[\sqrt{d}]$ je podkolebar $\mathbb{C}$

2) Množica $\mathbb{Q}[\sqrt{d}] = \{g+r\sqrt{d}; g, r \in \mathbb{Q}\}$ je podpolje $\mathbb{C}$ generirano z $\mathbb{Z}[\sqrt{d}]$

1) $\mathbb{Z}[\sqrt{d}] \subseteq \mathbb{C}$

testiraj za seštevanje, množenje, enote
za množenje:

$$(m+n\sqrt{d})(x+y\sqrt{d}) = mx + nyd + \sqrt{d}(nx + ym)$$

$$inde = 1 = 1 + 0 \cdot \sqrt{d}$$

$$\text{seštevanje: } (m+n\sqrt{d}) - (x+y\sqrt{d}) = (m-x) + (n-y)\sqrt{d}$$

2) invertiranje

$$\frac{m+n\sqrt{d}}{x+y\sqrt{d}} = \frac{(m+n\sqrt{d})(x-y\sqrt{d})}{x^2-yd} = \frac{c+e\sqrt{d}}{x^2-yd} \quad \checkmark$$

↑
konjugat

Automorfizmi $\mathbb{Z}[\sqrt{d}] = \{id; \sqrt{d} \mapsto \sqrt{d}\}$ 

Norma $N(x) = x \sigma(x)$

$$N(g+r\sqrt{d}) = (g+r\sqrt{d})(g-r\sqrt{d}) = g^2 - r^2d$$

$$3) \quad \forall x, y \in \mathbb{Z}[\sqrt{a}] : N(xy) = N(x)N(y)$$

$$N(xy) = xy\sigma(xy) = xy\sigma(x)\sigma(y) = x\sigma(x)y\sigma(y) = N(x) \cdot N(y)$$

$$4) \quad x \in \mathbb{Z}[\sqrt{a}] \text{ ist invertierbar} \Leftrightarrow N(x) = \pm 1$$

$$xy = 1 \Rightarrow N(xy) = N(1)$$

$$N(x)N(y) = 1$$

$$N(x) = \frac{1}{N(y)} \quad N(y) \in \mathbb{Z} \wedge N(x) \in \mathbb{Z}$$

$$\Rightarrow N(y) \in \{\pm 1\}$$

$$\Rightarrow N(x) \in \{-1, 1\}$$

5)

$$p \in \mathbb{P}; \quad N(x) = \pm p \Rightarrow x \text{ irreduzibel}$$

in $K(x) = \mathbb{F}_p$
 reicht da x irreduzibel ab abzulesen

$$x = ab$$

$$N(x) = N(a)N(b) \stackrel{!}{=} \pm p \Rightarrow N(a) = 1 \vee N(b) =$$

$$4) \quad \Leftarrow N(x) = 1, \quad x\sigma(x) \neq 1 \Rightarrow x^{-1} \notin \mathbb{Z}[\sqrt{a}]$$

5) umgekehrt
 hier da $N(x) \neq \pm 1 \Rightarrow a$ darstellbar.

$dc-1 \Rightarrow 1, -1$ sta edine obrnljive v $\mathbb{Z}[\sqrt{d}]$

$$x \text{ obrnljiv} \Rightarrow N(x) N(y) = \pm 1$$

$$N(a+b\sqrt{d}) = (a+b\sqrt{d})(a-b\sqrt{d}) = a^2 - b^2d = a^2 + b^2 \underset{1}{d}$$

$$\text{če je } b^2 > 1 \Rightarrow N > 1$$

$$\Rightarrow b = 0$$

a^2 mora biti 1 $\Rightarrow a = \pm 1$ oke edini;

morebiti
mimo de ste delilca to re je fi
rezultat

Pokaži da so $1+i$, $7+8i$, 3 nerazcepni v $\mathbb{Z}[i]$

Recimo da so razcepni

$$1+i = xy \quad N(xy) = x\sigma(x) y\sigma(y) = 2$$

$p \rightarrow$ nerazcepno prestavilo

$$N(7+8i) = 49+64 = \text{velik} = 113 \text{ preveliko}$$

~~ni preveliko~~ Nema je 9

$$3 = xy \quad N(xy) = x\sigma(x) y\sigma(y) = 9$$

vsej 2 morebiti 1

$$\cancel{N(x)} = N(\sigma(x)) = \cancel{\sigma(x)} \sigma(\sigma(x)) = \cancel{\sigma(x)} + \sigma(x)$$

$$\underline{N(x) = n} \Rightarrow \underline{n} \text{ razcepi}$$

$$N(x) = n = x\sigma(x)$$

6) Pāi šā vīse delīterjē elementa $2 \in \mathbb{Z}[i]$

$$x|2 \Leftrightarrow 2 = xy \quad \text{ar } n \nmid y$$

$$N(x, y) = 4 \quad N(x)N(y) = 4$$

$$x\sigma(x)y\sigma(y) = 4 = 2 \cdot 2 = xy\sigma(xy) = 2\sigma(xy)$$

$$\Rightarrow \sigma(xy) = 2 = \sigma(x)\sigma(y) \Rightarrow$$

$$N(x) = 2 \quad x = \pm 1 \pm i$$

$$x = \pm 2$$

$$x = \pm 2i$$

→ use mērītā

1)

$$a = 3+4i \quad \sigma = m^2 + n^2$$

$$b = 1-3i$$

$$|a| = 3+16 = 19$$

$$|b| = 1+9 = 10$$

magari della 10

$$a = kb + r$$

$$\frac{a}{b} = k + \frac{r}{b}$$

$$\frac{a}{b} \in \mathbb{Z}[i] \quad \frac{r}{b} \in \mathbb{Q}[i]$$

$$a = 3-4i$$

$$b = 1-3i$$

$$\frac{a}{b} = \frac{(3-4i)(1+3i)}{10} = \frac{3+12+i(3-4)}{10} = 1 + \frac{1}{2} + \frac{1}{2}i$$

$$\frac{1}{2}(1+i)(1-3i) = \frac{1}{2}(1+3+i(1-3)) = 2-i$$

$$a = b + 2-i \quad /: (2-i)$$

$$\frac{a}{2-i} = \frac{b}{2-i} + 1 \quad \Rightarrow$$

$$\frac{a}{2-i} - \frac{b}{2-i} = 1 \quad \Rightarrow \quad a-b = 2-i$$

2)

$$a=6$$

$$b=2+2\sqrt{5}$$

$$\mathbb{Z}[\sqrt{-5}]$$

$$N(a)=36 \quad N(b)=4+4\cdot 5=24$$

$$c \mid a, c \mid b \quad \text{Reemo de } d \mid a, d \mid b \Rightarrow d \mid c$$

$$c=2 \Rightarrow \frac{6}{2}=3 \quad \frac{b}{2}=1+\sqrt{5}$$

$$c=1+\sqrt{5} \Rightarrow \frac{a}{c}=1-\sqrt{5}, \quad \frac{b}{c}=2$$

$$J(c)=1+5=6$$

$$\begin{array}{cc} 6 \cdot 2 \cdot 3 & 6 \cdot 2 \cdot 3 \\ \parallel & \parallel \\ 6^2 & 6 \cdot 4 \\ \parallel & \parallel \\ 1 & 1 \end{array}$$

$$N(\gcd(a,b)) = \gcd(N(a), N(b)) = \gcd(36, 24) = 12 = x^2 \cdot 5y^2$$

$$\begin{array}{c} y^2 \in \{1, 2\} \\ \downarrow \\ x^2 = 3 \quad x^2 = x \\ \text{Take negative} \end{array}$$

3) UFD $\Rightarrow$ gcd istnieje

$$a = \prod_{i \in I} p_i^{k_i}$$

$$b = \prod_{i \in I} p_i^{l_i}$$

$$d = \gcd(a, b) = \prod_{i \in I} p_i^{\min\{k_i, l_i\}}$$

$$d|a \wedge d|b$$

Najm $c = \prod_{i \in I} p_i^{m_i} \quad c|a \wedge c|b$

$$a = c\alpha \quad \alpha = \prod p_i^{n_i}$$

$$\text{Dla } a = \prod p_i^{n_i + m_i} = \prod p_i^{k_i} \Rightarrow m_i \leq k_i$$

$\text{zatem } m_i < l_i \Rightarrow \nexists i$

$$c|d$$

$\mathbb{Z}[\sqrt{-2}]$ additief

a) $\forall a, b \neq 0 \exists g, r \in K. a = gb + r \wedge r = 0 \vee \sigma(r) < \sigma(b)$

b) $\sigma(a) \leq \sigma(a, b) \forall a, b \in K - \{0\}$

$$\sigma: m + \sqrt{-2}n = m^2 + 2n^2$$

vens de b volgt

$a, b \in \mathbb{Z}[\sqrt{-2}]$ plus $a = x + \sqrt{-2}y \quad b = m + \sqrt{-2}n$

$$\frac{a}{b} = g_1 + g_2 \sqrt{-2}$$

$$a = ([g_1] + [g_2]\sqrt{-2})b + ([g_1 - [g_1]] + ([g_2 - [g_2]]\sqrt{-2}))b$$

DN:

Naj bo $K[X]$ UFD

Polem ime polinom n ^{ek} endice delitev
Torej je tudi $K[X]$ UFD

Naj bo K UFD

Naj bo $p(x)$ poljuben polinom

$$\begin{aligned} \text{Razcepimo } p(x) &= \prod_{k=1}^r \left(\sum_{i=1}^n a_{i,k} x^i \right) = \\ &= \prod_{k=1}^r (a_{n,k} x^n + \dots + a_{0,k}) = x^{nr} \underbrace{\left(\prod_{k=1}^r a_{n,k} \right)} + \dots \end{aligned}$$

$$\text{cont}(fg) = \text{cont}(f) \text{cont}(g)$$

$$p \mid \text{cont}(fg) \Rightarrow \gcd(a_0, \dots, a_n) = p$$

$$fg = 0 \vee K_p[X]$$

$$\Rightarrow p=0 \vee g=0$$

$$(ax+b)(cx+d) = acx^2 + x(cb+ad) + bd$$

$$\Rightarrow p \mid \text{cont } f \vee p \mid \text{cont } g$$

Remede s' rapere FOX

plus les zéros et $a = \rho(z) \in K \setminus T$

Nefta shells don't (at) her at = 12 / out

$$\text{conf}(af) = \text{cont}(\underline{g}) \text{cont}(p) = a \text{ conf}$$

$$= a \cdot \text{end} : \text{Ov}$$

endless changes delightfully

$$a | \text{cont}(p_2) \Rightarrow \frac{p}{x} \cdot \frac{y^2}{y} = \frac{fa}{a} = f$$

$$\begin{array}{cc} \boxed{x} & \boxed{y} \\ v_1 & v_2 \end{array}$$

~~22~~

$$f|_{p_2} \Rightarrow f|_p \vee f|_g$$

$$kf = p_2 \quad \text{cont} f \text{cont} k = \text{cont } p \text{cont} g$$

$p_2 \text{ max } \Rightarrow f/p \vee +/g_2$

$\Rightarrow \forall p, q \in \mathcal{P} \Rightarrow p \sim q$

2) $K[X]$ glavni ideal $\Leftrightarrow K$ je polje

$$a \in K - \{0\}$$

$$\exists b \text{ s.t. } ab = 1$$

$$(a) \text{ je ideal } (a) = \{af; f \in K[X]\}$$

↑ $1+\sqrt{-3}$ nerazcepen v $\mathbb{Z}[\sqrt{-3}]$,
ampak ni pračlen

$1+\sqrt{-3}$ nerazcepen

Recimo da je $(1+\sqrt{-3}) = \alpha \cdot \beta$

$$N(\alpha \cdot \beta) = N(\alpha) \cdot N(\beta) = 1+3=4$$

$$N(\alpha) = N(\beta) = \pm 2$$

Če bi bila 1 bi bila delilnik

$$N(a+\sqrt{-3}b) = a^2 + 3b^2 = \pm 2$$

$$b=0$$

$$a^2 = \pm 2 \text{ neobstaja}$$

Ni pračlen

$$\cancel{(a+b\sqrt{-3})}(1+\sqrt{-3}) = \cancel{a+b\sqrt{-3}+a\sqrt{-3}+b} =$$

Vsek element deli svojo normo

$$(1+\sqrt{-3})(1-\sqrt{-3}) = 4 = 2 \cdot 2$$

$$\underline{\underline{(1+\sqrt{-3}) \nmid 2}}$$

$$\text{Recimo da } k(1+\sqrt{-3}) = 2 \quad N(2) = 4 \Rightarrow$$

$$N(k) = \pm 1$$

$$\Rightarrow k = \pm 1$$

Domena relacje

$$\omega = \frac{(1+\sqrt{-19})}{2} \quad \mathbb{Z}[\omega] = \{a+b\omega : a, b \in \mathbb{Z}\}$$

$$N(a+b\omega) = |a+b\omega|^2 = a^2 + ab + 5b^2$$

2 in 3 rozdzępnę

$$N(2) = 4$$

$$2 = ab ; a, b \text{ nieobr.} \Rightarrow N(a) = N(b) = 2$$

$$a = a + b\omega$$

$$N(a) = a^2 + ab + 5b^2 = 2$$

$$b \neq 0 \Rightarrow 5b^2 > 2 \Rightarrow N(a) > 2 \quad \times$$

$$N(3) = 9 \quad 9 = x \cdot y \quad 9 = 3 \cdot 3 \Rightarrow N(x) = N(y) = 3$$

$$N(x) = 3 \quad x = a + b\omega$$

$$a^2 + ab + 5b^2 = 3$$

$$b \neq 0 \Rightarrow 5b^2 > 3 \Rightarrow N(x) > 3 \quad \times$$

$$N(x) = 1 \Rightarrow x \in \underline{\underline{\{1, -1\}}}$$

$$a^2 + ab + 5b^2 = 1$$

$$b \neq 0 \Rightarrow 5b^2 > 1 \quad \times \Rightarrow b = 0$$

$$a^2 = 1 \Rightarrow a \in \{1, -1\} \Rightarrow x \in \{1, -1\}$$

Moduli

$K = \{ \text{zagrane tritelne matrice nad } F \}$

$M = \begin{bmatrix} F \\ F \end{bmatrix}$ je lev K -modul
iscemo podmodule od M

$$KM \subseteq M$$

$$N = \{ \lambda \begin{bmatrix} u \\ v \end{bmatrix} ; \lambda \in F \} \text{ za } u, v \in F$$

$$\begin{bmatrix} a & b \\ c & c \end{bmatrix} \cdot \lambda \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} au + bv \\ cu \end{bmatrix} \in N$$

$$\lambda au + \lambda bv = \mu u$$

$$\lambda cv = \mu v \Rightarrow \lambda c = \mu$$

$$\lambda au + \lambda bv = \lambda cu$$

$$au - cu + bv = 0$$

$$bv = (c - a)u$$

DW najpierw

$\mathbb{Z}[\omega]$ n: całkowity

$$(\exists \delta: \mathbb{Z}[\omega] - \{0\} \rightarrow \mathbb{N} \cup \{0\})$$

$$\omega = gx + r \Rightarrow r \in \{1, -1, 0\}$$

$$r=0 \Rightarrow \omega = gx \Rightarrow x \text{ asoc'ion z } \omega$$

$$N(x) = N(\omega) = 5$$

$$2 = g'x \pm 1 \Rightarrow x \in \{2, \pm 3\} \Rightarrow N(x) \in \{4, 9\}$$

$$r=1 \Rightarrow \omega = gx + 1$$

~~*~~

$$-1 + \omega = gx \Rightarrow 1 - 1 + 5 = N(g)N(x) = 5$$

$$r=-1 \Rightarrow 1 + \omega = gx \Rightarrow 7 = N(g)N(x) \Rightarrow N(x) = 5 \quad \times$$

$$\Rightarrow N(x) = 7 \quad \times$$

$\mathbb{Z}[w]$ je glavni

$$(\forall \alpha \in \mathbb{C} - \mathbb{Z}[w], \exists p, q \in \mathbb{Z}[w], 0 < |p\alpha - q| < 1) \\ \Rightarrow \mathbb{Z}[w] \text{ glavnikolobar}$$

Mej $b \in I$ poljubno: ideal. Mej $b \in I$ tak da

$$\forall x \in I, N(x) \geq N(b)$$

$$\text{Mej } b \in I \text{ poljubno } \alpha = \frac{a}{b} \quad a = \alpha b$$

$$\alpha \in \mathbb{Z}[w] \Rightarrow b|a \quad \forall a \in I \Rightarrow I = (b)$$

$$\alpha \in \mathbb{Z}[w] \text{ in } \{p\alpha + q\} \Rightarrow 0 < |p\frac{a}{b} - q|^2 < 1 \quad / \cdot |b|$$

$$0 < \underbrace{|p\alpha - q|}_{\in I}^2 < \underbrace{|b|^2}_{\in I} \quad \times$$

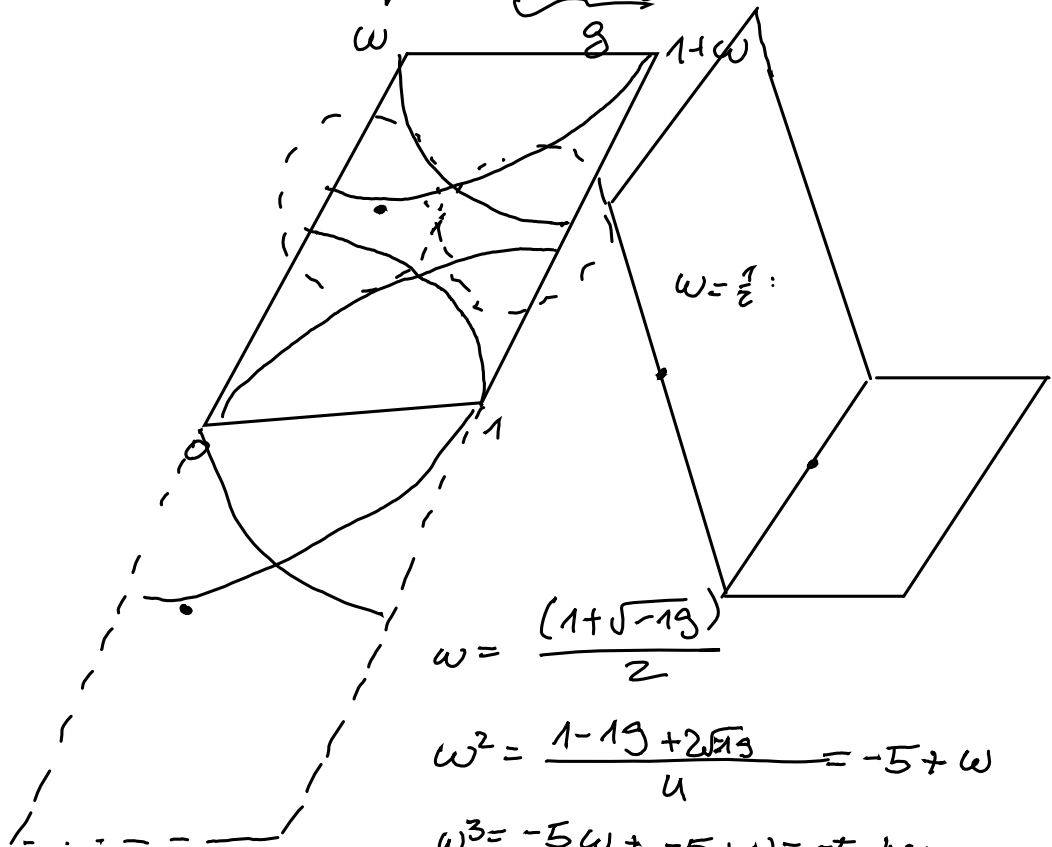
$$\forall \alpha \in \mathbb{C} - \mathbb{Z}[\omega], \exists \beta, \gamma \in \mathbb{Z}[\omega], 0 < |\rho\alpha - \gamma| < 1$$

$\alpha \in \mathbb{C} - \mathbb{Z}[\omega]$ polzunten

$$\alpha = a + b\omega \quad a, b \in \mathbb{R}$$

$$\rho\alpha - \gamma = \rho(a + b\omega) - \gamma = \rho a - \gamma + \rho b\omega$$

$$\rho\alpha - \gamma = \rho(\beta + \rho(\dots)) - \gamma$$



DN ze teche:

$$d = 4k + 1$$

$$(a + b\sqrt{d})(a - b\sqrt{d}) = a^2 - db^2 = 4(2 - k)$$

$$b^2 = 1:$$

$$a^2 - 4k - 1$$

$$a = 3$$

$$8 - 4k = 4(2 - k)$$

$$b^2 = 1$$

$$a^2 = 9$$

$$\{(x-1)^k \mid k \geq 0\} = \sum_{n=0}^{\infty} a_n x^n = 0$$

That is ideal

$$\sum_{n=0}^k a_n x^n$$

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 : (x-1) = a_n x^{n-1} + a_{n-1} x^{n-2} + \dots + a_1$$

$(a_n + a_{n-1}) x^{n-1} + \dots + a_1$ we get
 edge case $n=0$
 and prep.

$$ax - a = a(x-1) \quad \checkmark$$

Ques in the pi $\exists B \ni \{f_1, \dots, f_g\}$

Let α don't like α ~~per se~~

$g_1 f_1 + \dots + g_n f_n$

$\sqrt{f_1 + \dots + f_n}$

$\sqrt{p_1 f_1 + \dots + p_n f_n} \leq \sqrt{p_1 + \dots + p_n} \sqrt{f_1 + \dots + f_n}$

$\exists h_1 f_1 + \dots + h_n f_n$

$$M = \mathbb{Z} \mathbb{Z}_{12}$$

$$\mathbb{Z}_2 = \{0, 6\} \quad \text{even}$$

$$\mathbb{Z}_3 = \{0, 4, 8\} \quad \text{multiple}$$

$$\mathbb{Z}_4 = \{0, 3, 6, 9\} \supseteq \mathbb{Z}_2$$

$$\mathbb{Z}_6 = \{0, 2, 4, 6, 8, 10\} \supseteq \mathbb{Z}_2, \mathbb{Z}_3$$

$$3) \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_{12}), \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_3, \mathbb{Z}_4), \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_{12}, \mathbb{Z}_{12})$$

$$\mathbb{Z} \rightarrow \mathbb{Z}_{12}$$

$$0 \mapsto 0 \quad \text{1 naturals left}$$

$$1 \mapsto \text{ker} \Rightarrow \text{image is } 12 \text{ mod } 12$$

K klobber, kM modul $\Rightarrow$

$$\operatorname{Hom}(K, M) = \{x \mapsto xm : m \in M\} \cong {}_K M$$

$$\mathbb{Z}_3 \rightarrow \mathbb{Z}_4$$

$$0 \mapsto 0$$

1.

$$1 \mapsto 1$$

$$2 \mapsto 2$$

$$3 \mapsto 0$$

$$4 \mapsto 1$$

2.

$$1 \mapsto 2$$

$$2 \mapsto 1$$

$$3 \mapsto 0$$

$$4 \mapsto 2$$

3.

$$n \mapsto 0$$

$$\mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$$

$$0 \mapsto 0$$

$$1 \mapsto a$$

$$n \mapsto na, \text{ mod } 12$$

K kolčar. K enostaven $\Leftrightarrow K$ brez
 $\Rightarrow$ Rečnoda je enostaven, a enojni
 a obratno

$$K_a = K \Rightarrow \exists k \in K. ka = 1$$

$\Leftarrow$ je potrebno

Vaje za rezej

1. Nakleži M K -modul $M \neq \{0\}$ enostaven $\Leftrightarrow M = K_m$

$\forall m \in M$

Naj bo M enostaven. Edina podmodula sta $\{0\}$ in M

$\Rightarrow$ Naj bo $m \in M$ poljuben K_m je podmodul!

$$\alpha, \beta \in K \quad \alpha k_m - \beta h_m = (\alpha k - \beta h)_m \in K_m$$

$$\Leftarrow M = K_m \text{ za } \forall m \in M.$$

Naj bo N podmodul.

$$\forall n \in N. K_n = M \text{ torej } N = M$$

2. naloge

levi: k modul M enostaven $\Leftrightarrow M \cong K/I$, kje r je
 $\perp$ k maksimalen levi ideal

$\Rightarrow$ Naj bo M enostaven, $a \in M$

$\varphi: K \rightarrow M$ φ surjektiv po prejšnji nalogi
 $k \mapsto ka$

$\ker \varphi = \{k; ka = 0\} = {}_K I$ maksimalen

$I \subsetneq J \subsetneq K$

$K \rightarrow M$

$J \xrightarrow{\text{sur}} \varphi(J)$

$I \rightarrow 0$

$\ker J \neq \ker \varphi$
 $\downarrow$
 $0 \subsetneq \varphi(J) \subsetneq M$
 $\Rightarrow J = I \vee J = K$
 $\varphi_*(J) = M \Rightarrow m = ja$
 $\Rightarrow j = ja \Rightarrow j(1-a) = 0$
 $1-a \in \ker \varphi$
 $\Rightarrow 1-a \in J$
 $(1-a) + a = 1 \in J$
 $\Rightarrow J = K$
 $\star$

$\Leftarrow$ Naj bo ${}_K M \cong K/I$; I maksimalen levi ideal

$0 < N < M$

$\uparrow$

$I < J < K$

φ surj

$J = \varphi^*(N)$

$N \subseteq \varphi_*(J) = \varphi_*(\varphi^*(N)) \Rightarrow \varphi_*(J) = N$

$K/I \rightarrow M$

$1+I \mapsto e \in M - \{0\}$

$x+I = x(1+I) \mapsto xe$

$\varphi: K \rightarrow M$

$k \mapsto ke$

Menostaven

$I = J \vee J = K$

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

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$\downarrow$

$0 < \varphi_*(J_2) = \varphi_*(J_1) < M$

$\therefore \varphi(J_2)$

$0 = \frac{\varphi_*(J_1)}{\varphi_*(J_2)} < M/\varphi(J_2)$

??

У нас не было
вспомог

Od zadnjic:

$$\text{End}(M \oplus N) \cong \begin{bmatrix} \text{End}(M) & \text{Hom}(N, M) \\ \text{Hom}(M, N) & \text{End}(N) \end{bmatrix}$$

1) določi kalobarje endomorfizmov

$$\text{End}_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_3), \text{End}_{\mathbb{Z}}(\mathbb{Z}_6 \oplus \mathbb{Z}_3)$$

$$\text{End}_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_3) = \begin{bmatrix} \text{End } \mathbb{Z}_2 & \text{Hom}(\mathbb{Z}_3, \mathbb{Z}_2) \\ \text{Hom}(\mathbb{Z}_2, \mathbb{Z}_3) & \text{End } \mathbb{Z}_3 \end{bmatrix}$$

$$\text{End } \mathbb{Z}_2: \begin{matrix} 0 \mapsto 0 & 1 \mapsto 0 \\ 1 \mapsto 1 & 0 \mapsto 1 \end{matrix} \cong \mathbb{Z}_2$$

$$\text{Splošno: } \text{End } \mathbb{Z}_p \cong \mathbb{Z}_p \Rightarrow \mathbb{Z}_3 \cong \mathbb{Z}_3$$

$$\text{End}(\mathbb{Z}_2 \oplus \mathbb{Z}_3) \cong \begin{bmatrix} \mathbb{Z}_2 & 0 \\ 0 & \mathbb{Z}_3 \end{bmatrix} \xrightarrow{1 \mapsto \text{red demerita 3 ne ak steje in}} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$$

$$\text{End}(\mathbb{Z}_6 \oplus \mathbb{Z}_3) : \text{End}(\text{End}(\mathbb{Z}_2 \oplus \mathbb{Z}_3) \oplus \mathbb{Z}_3) =$$

$$= \begin{bmatrix} \text{End}(\mathbb{Z}_2 \oplus \mathbb{Z}_3) & \text{Hom} \\ \text{Hom} & \text{End } \mathbb{Z}_3 \end{bmatrix} = \begin{bmatrix} \mathbb{Z}_6 & \text{Hom} \\ \text{Hom} & \mathbb{Z}_3 \end{bmatrix}$$

$$\text{Hom}(\mathbb{Z}_6, \mathbb{Z}_3) \cong_{\mathbb{Z}} \text{Hom}(\mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z}_3) \cong$$

$$\cong \underbrace{\text{Hom}(\mathbb{Z}_2, \mathbb{Z}_3) \oplus \text{Hom}(\mathbb{Z}_3, \mathbb{Z}_3)}_0 \cong \mathbb{Z}_3$$

$$\text{End}(\mathbb{Z}_6 \oplus \mathbb{Z}_3) = \begin{bmatrix} \mathbb{Z}_6 & \mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix}$$

$$\begin{matrix} 1 \mapsto 2 \\ \downarrow \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{matrix} \begin{matrix} \uparrow \\ \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \end{matrix} \stackrel{?}{=} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \text{ se ne izide}$$

$$x \mapsto (2, x \bmod 3)$$

$$\begin{bmatrix} 0 & \mathbb{Z}_3 \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_3 \rightarrow \mathbb{Z}_6 \\ 0 & 1 \mapsto (1, 0) \mapsto 4 \\ & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \mathbb{Z}_6 \cong \mathbb{Z}_2 + \mathbb{Z}_3 \rightarrow \mathbb{Z}_3 \\ (1, 0) \mapsto 2 \\ x \mapsto 2x \bmod 3 \end{bmatrix}$$

$$= \begin{bmatrix} x \mapsto 2x \mapsto (2x, 0) = 4x \\ \cancel{2 \mapsto 8 \bmod 3 = 2} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\updownarrow \mathbb{Z}_6 \rightarrow \mathbb{Z}_3$$

$$\begin{bmatrix} 0 & x \mapsto 4x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ x \mapsto 2x \\ \mathbb{Z}_3 \rightarrow \mathbb{Z}_6 \end{bmatrix} \begin{bmatrix} x \mapsto 8x & 0 \\ 0 & 0 \end{bmatrix}$$

$$k: \mathbb{Z}_3 \rightarrow \mathbb{Z}_6$$

$$\mathbb{Z}_6 \rightarrow \mathbb{Z}_3$$

$$1: 1 \mapsto (1, 0) \mapsto 4$$

$$k: 1 \mapsto k \bmod 3$$

$$2: 1 \mapsto (2, 0) \mapsto 2$$

$$k: 1 \mapsto (k, 0) \mapsto \dots$$

Hilfsatz:

$$\mathbb{Z}_{18} = \mathbb{Z}_6 \oplus \mathbb{Z}_3 = \mathbb{Z}_2 \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_3)$$

$$\text{End}(\mathbb{Z}_6 \oplus \mathbb{Z}_3) = \begin{bmatrix} \mathbb{Z}_2 & \text{Hom}(\mathbb{Z}_3 \oplus \mathbb{Z}_3, \mathbb{Z}_2) \\ \text{Hom}(\mathbb{Z}_2, \mathbb{Z}_3 \oplus \mathbb{Z}_3) & \text{End}(\mathbb{Z}_3 \oplus \mathbb{Z}_3) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbb{Z}_2 & 0 & \text{Hom}(\mathbb{Z}_2, \mathbb{Z}_3) \oplus \text{Hom}(\mathbb{Z}_2, \mathbb{Z}_3) \\ 0 & \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathbb{Z}_2 & 0 & 0 \\ 0 & \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix} \cong$$

$$\underline{\underline{\mathbb{Z}_2 \oplus M_2(\mathbb{Z}_3)}}$$

Bimoduli

$$K^{\circ p} = \{ x^{\circ p} \mid x \in K \} \quad x^{\circ p} \cdot y^{\circ p} = (yx)^{\circ p}$$

$$\text{Grupa: } G \cong G^{\circ p} \\ x^{\circ p} = x^{-1}$$

${}_K M \cong M_{{}_K K^{\circ p}}$ vsak levi modul lahko predstavimo kot desni K modul

Bimodul je ${}_K M_S$, povezuje prava asociativnost: K, S kolobarja ${}_K M$ je levi K -modul M_S je desni S -modul zakon

$$x \in K, y \in S$$

$$m \in M \quad \underbrace{(x \cdot m)}_{\in M} y = x(m \cdot y)$$

Primeri

1) ${}_K K_K$ je bimodul

2) ${}_K M_{{}_K K^{\circ p}}$ ni bimodul (razen če K komut.)

$$\begin{aligned} x(m \cdot y^{\circ p}) &= x y m \\ m \cdot x^{\circ p} &= x m \\ (x m) \cdot y^{\circ p} &= y x m \end{aligned}$$

3) ${}_K M_{\text{End}(M)^{\circ p}}$ je bimodul

$$(x m) \cdot \varphi^{\circ p} = \varphi(x m) = x \varphi(m) = x(m \cdot \varphi^{\circ p})$$

2) Naj bosta ${}_K M_S$ in ${}_K N_R$ bimodule (K, S, R kolobarj)

Premisi, da imajo homomorfizmi: $M \rightarrow N$ reverse
 strukturo (S, R) -bimodule $\xleftarrow{\text{homomorfizir. kot levi } K\text{-moduli}}$
 zgubi strukturo K module

$${}_S \text{Hom}(M, N)_R$$

$$s \in S \quad s\varphi := m \mapsto \varphi(ms)$$

$$r \in R \quad \varphi r := m \mapsto \varphi(m)r$$

$\text{Hom}(M, N)$ je levi S modul

$$(s+t)\varphi = (m \mapsto \varphi(m(s+t))) = (m \mapsto \varphi(ms) + m \mapsto \varphi(mt)) \\ \parallel \\ \varphi(ms) + \varphi(mt) = s\varphi + t\varphi$$

$$1 \cdot \varphi = (m \mapsto \varphi(m)) = \varphi$$

$$S(\varphi + \psi) = \checkmark$$

$$\forall m. (st)\varphi(m) = \varphi(mst) = \varphi((ms)t) =$$

$$= (t\varphi)(ms) = s(t\varphi(m)) = s(t\varphi)(m)$$

$$\uparrow \\ \text{ali je to vredno } \underline{(t\varphi)(xm) = x((t\varphi)(m))}$$

$$(t\varphi)(xm) = \varphi(xm + t) = \varphi(x(m+t)) = x\varphi(mt)$$

Podobno za desni R module

$$S(\varphi r)(m) = (\varphi r)(ms) = \varphi(ms)r = (s\varphi(m))r$$

Lev K -modul ${}_K M$. **Dual** je $M_K^* = \text{Hom}({}_K M, {}_K K)$
 K je desna bimodul, je tudi $M_K^* \overset{\uparrow}{\text{Funktional}}$ desni K -modul

$$M = \begin{bmatrix} F & F \\ F & F \end{bmatrix}$$

3) Določiti $(M_2(F) M)^*$

$$M_{M_2(F)}^* = \text{Hom}(M_2(F) M, M_2(F)_{M_2(F)}) \cong$$

$$\begin{bmatrix} F & F \\ F & F \end{bmatrix} \cong \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix} \cong M \oplus M$$

$$\cong \text{Hom}(M, M \oplus M) = \text{End}(M) \oplus \text{End}(M) \cong F^2$$

$\uparrow \nearrow$
 To sta obsega (shukava lina)

$$M = \begin{bmatrix} F & \\ & F \end{bmatrix} \quad M \stackrel{?}{=} \begin{bmatrix} F & F \\ & F \end{bmatrix}$$

$$K = \begin{bmatrix} F & F \\ F & F \end{bmatrix} \quad K \cong M \oplus M \text{ ristična + vristica}$$

$$\begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix} \oplus \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix}$$

struktura
desnega
K-modula

$$M^* = \text{Hom}(M, K) \cong \text{Hom}(M, M) \oplus \text{Hom}(M, M) \cong F \oplus F$$

↑
ni desni K modul.

izgubimo strukturo modula

$$(*) \quad F \cong \text{Hom}(M, M) = \text{End } M$$

End(M) obseg, če M enostaven

$$F \cong \text{End}(M)$$

2-dim vektorski prostor

$$\geq \text{V posebenem: } \varphi \in \text{Hom}_K(M, M) \leq \text{Hom}_F(M, M) \cong M_2(F)$$

$$\varphi(Ax) = A\varphi(x) \quad \varphi = B \Rightarrow$$

$$BAx = ABx \quad \forall A \in K \quad \forall x \in M \Rightarrow BA = AB \Leftrightarrow \forall A \in K.$$

$$B \in Z(M_2(F)) = F \cdot I \quad \begin{matrix} \swarrow \text{diagonalne} \\ \searrow \text{matrice} \end{matrix}$$

≤ očitno

≅ F

$$(**) \quad M_K^* \text{ is } (F \oplus F)_K$$

$$\varphi \in \text{Hom}(M, K)$$

$$\varphi(m) = (\varphi_1(m), \varphi_2(m)) = (\varphi(m) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \varphi(m) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

← očitno

$$\varphi(m) = \begin{bmatrix} \varphi_1(m) \\ \varphi_2(m) \end{bmatrix}$$

$$\varphi(Am) = \begin{bmatrix} \varphi_1(Am) \\ \varphi_2(Am) \end{bmatrix}$$

$$A\varphi(m) = \begin{bmatrix} A\varphi_1(m) \\ A\varphi_2(m) \end{bmatrix} \quad \varphi_{1,2} \text{ skalarne vektorske identitete}$$

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \quad \varphi \cdot A = (m \mapsto \varphi(m) \cdot A) \quad \begin{matrix} \swarrow M^* \\ \searrow \text{dano po definiciji} \end{matrix}$$

$$[\varphi_1, \varphi_2] \cdot A = \varphi_1 \cdot a + \varphi_2 \cdot b + \varphi_1 \cdot c + \varphi_2 \cdot d = \varphi_1(a+c) + \varphi_2(b+d)$$

↑ kako sklopamo danje

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \quad \varphi(m) = [\varphi_1(m), \varphi_2(m)] \quad m = \begin{bmatrix} x \\ y \end{bmatrix}$$

↑ skalarje

$$= \begin{bmatrix} \lambda_1 m & \lambda_2 m \end{bmatrix} = \begin{bmatrix} \lambda_1 x & \lambda_2 x \\ \lambda_1 y & \lambda_2 y \end{bmatrix}$$

$$\varphi(m) \cdot A = \begin{bmatrix} a\lambda_1 x + b\lambda_2 x & c\lambda_1 x + d\lambda_2 x \\ a\lambda_1 y + b\lambda_2 y & c\lambda_1 y + d\lambda_2 y \end{bmatrix} =$$

$$= \begin{bmatrix} x(a\lambda_1 + b\lambda_2) & x(c\lambda_1 + d\lambda_2) \\ y(a\lambda_1 + b\lambda_2) & y(c\lambda_1 + d\lambda_2) \end{bmatrix}$$

$$\varphi = [\lambda_1 \quad \lambda_2] \quad \varphi \cdot A = [\lambda_1 a + \lambda_2 b, \quad c\lambda_1 + d\lambda_2]$$

$$\text{Lažje: } F \oplus F \rightarrow \text{Hom}(M, K)$$

$$[\lambda, \mu] \mapsto \left(\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix} [\lambda, \mu] \right)$$

$$(\varphi \cdot A) \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\left(\begin{bmatrix} x \\ y \end{bmatrix} [\lambda, \mu] \right)}_{\varphi} A = \begin{bmatrix} x \\ y \end{bmatrix} ([\lambda, \mu] A)$$

↑

1) Dokazi dual $\mathbb{Z}_n^*$

$$= \text{Hom}(\mathbb{Z}_n, \mathbb{Z}) = 0$$

$$1 \mapsto x \quad \text{red } 1 \mid \text{red } x \Rightarrow x = 0$$

2) K cel kolobar (brez del.ni~~ca~~, komut)

M K -modul definiramo

$$\text{tor}(M) = \{m \in M; xm = 0, x \in K - \{0\}\}$$

$$\text{tor}\left(\bigoplus_{n=2}^{\infty} \mathbb{Z}_n\right) = \bigoplus_{n=2}^{\infty} \mathbb{Z}_n; \text{ a mak neabstaja element, ki bi}$$

unicil vse

Dokazi da je $\text{tor}(M) \subseteq M$ podmodul in

$$\varphi: M \rightarrow N$$

$$\varphi_*(\text{tor}(M)) \subseteq \text{tor}(N)$$

a) $xm = 0 \quad ym = 0 \quad m, n \in M \quad x, y \in K$

$$\underbrace{xy}_{\neq 0}(m+n) = xym + x \underbrace{yn}_0 = yxm = 0$$

$$\alpha \in K \quad xm = 0$$

$$x \alpha m = \alpha xm = 0$$

b) φ homomorfizem

$$x\varphi(m) = \varphi(xm) = 0 \rightarrow m \in \text{tor}(M) \Rightarrow \varphi(m) \in \text{tor}(N)$$

$$\Rightarrow \varphi_*(\text{tor}(M)) \subseteq \text{tor}(N)$$

$$\text{je podmodul: } x(\alpha\varphi(m) + \beta\varphi(n)) =$$

$$\underbrace{\alpha y \varphi(xm)}_0 + \beta \underbrace{x \varphi(n)}_0 = 0$$

$$\text{tor}(M/\text{tor}(M)) = 0 \quad \text{DN}$$

$$\chi(m + \text{tor} M) = \chi m + \text{tor} M$$

$$\underbrace{= 0} \Rightarrow m \in \text{tor} M \rightarrow m + \text{tor} M = 0$$

$$\neq 0 \Rightarrow m \notin \text{tor} M$$

3. M je prost $\Rightarrow \text{tor}(M) = 0$

$$\varphi: M \xrightarrow{\cong} \bigoplus_{i \in I} K$$

$$\ell(\text{tor} M) \leq \text{tor}(\bigoplus_{i \in I} K)$$

$$\nexists \text{ inj} \Rightarrow \text{dovolj je dokazat } \text{tor}(\bigoplus_{i \in I} K) = 0$$

(Dovolj je inj vlezitev, ne izzame (izem))

$$\bigoplus_{i \in I} K = \{ (x_i)_{i \in I} ; x_i \in K \mid x_i = 0 \text{ za vse razen konocno mnoge} \}$$

$$(x_1, x_2, \dots, x_i, \dots)$$

$$(x_i)_{i \in I} \in \text{tor}(\bigoplus K)$$

$$\gamma(x_i)_{i \in I} = (\gamma x_i)_{i \in I} \Leftrightarrow \gamma x_i = 0 \quad \forall i$$

nevelja za noben element razen 0

$$\Rightarrow x_i = 0 \quad \forall i$$

4) Ali je $\mathbb{Z}_n$ prost $\mathbb{Z}$ -modul?

NE, ker ima torzijsko

Izrek: M kanonsko generiran $\mathbb{Z}$ -modul.

M prost $\Leftrightarrow M$ torzijsko prost

5) Pokaži

$\mathbb{Z}[\mathbb{Q}]$ ni prost $\mathbb{Z}$ -modul

$$\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$$

$$(cb) \frac{a}{b} - (ad) \frac{c}{d} = 0 \quad \text{netrivialna linearna kombinacija}$$

1: jlahke baze $\Rightarrow$

$$\mathbb{Q} = \left(\frac{a}{b} \right) \quad \frac{1}{b^2} \text{ ni delilek } \propto \frac{a}{b}$$

Ali je $(\mathbb{Q}^*, \cdot)$ prost $\mathbb{Z}$ modul

$$-1 \in \mathbb{Q}^* \quad (-1)(-1)=1 \Rightarrow (-1) \in T(\mathbb{Q}^*)$$

$\Rightarrow$ ima torzijsko, torej ni prost

kar pa $(\mathbb{Q}_+, \cdot)$

$$\sum \left(\frac{a_i}{b_i} \right)^{n_i} > 0 \neq 0$$

Eksaktna zaporedja

kratke = kvocienti

$$0 \xrightarrow{\phi} L \xrightarrow{\psi} M \xrightarrow{\chi} N \xrightarrow{\theta} 0$$

$$\text{im } \phi = \ker \psi \quad \text{im } \chi = \ker \theta = N$$

$$\phi = \text{im } 0 = \ker \phi$$

ψ injektivna

χ surjektivna

$$M / \phi(L) \cong N$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

$$x \mapsto 2x \quad x \mapsto x \bmod 2$$

inj



odina možna preostanka

$$\ker \psi = 2\mathbb{Z} = \text{im } \phi$$

$$0 \xrightarrow{\text{inj}} \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{T} \xrightarrow{\text{sur}} 0$$

enota krožnice

$$n \mapsto \begin{bmatrix} n \\ x \end{bmatrix} \mapsto 1$$

$$x \mapsto e^{2\pi i x}$$

P je **projektiv** če $\forall$ eksaktno zaporedje $0 \rightarrow L \rightarrow M \rightarrow P \rightarrow 0$ razpade

kratko eksaktno zaporedje razpade če je ekvivalentno $0 \rightarrow L \rightarrow L \oplus N \rightarrow N \rightarrow 0$

P je projektiven $\Leftrightarrow 1) \Leftrightarrow 2)$

$$\Leftrightarrow 1) \quad \begin{array}{ccc} & & P \\ \exists g, \text{---} & \nearrow & \downarrow hf \\ A & \xrightarrow{\text{sur}} & B \rightarrow 0 \end{array}$$

$\Leftrightarrow 2) P$ je direktni sumand prostega modula $K^{(I)}$

$$\exists N. P \oplus N \cong K^{(I)} = \bigoplus_{i \in I} K$$

↑
kolobar

1. naloge

7.12

$\begin{bmatrix} F \\ F \end{bmatrix}$ je projektiven in ne prost nad $M_2(F)$

$$2) M_2(F) \cong \begin{bmatrix} F \\ F \end{bmatrix} \oplus \begin{bmatrix} F \\ F \end{bmatrix} \cong \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$$

$\nwarrow$ zunanji $\nearrow$ notranji

$$\begin{bmatrix} x & z \\ y & w \end{bmatrix} \mapsto \left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix} \right) \mapsto \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & z \\ 0 & w \end{bmatrix}$$

je osten izometričnem

$\Rightarrow$ je projektiven

Recimo da je prost, potem

$$\begin{bmatrix} F \\ F \end{bmatrix} = \begin{cases} M_2(F) = \dim 4 \\ M_2(F) \oplus M_2(F) = \dim 8 \end{cases} \quad \begin{matrix} \text{vse dimenzije} \\ \text{so prave} \end{matrix}$$

$\Rightarrow \begin{bmatrix} F \\ F \end{bmatrix}$ ni prost

2. način:

$P \text{ prost} \Rightarrow P \text{ oz } P^2 \text{ prost}$

$P \xrightarrow{\varphi} M$ homomorfizem

$$\varphi(\text{tor}(P)) \subseteq \text{tor}(M)$$

$$2) \quad P \text{ proj. } \xRightarrow{(K \text{ cel})} P \text{ tarz. prost}$$

Velja: $\varphi: P \rightarrow M$ hemo.
 $\varphi(\text{tor}(P)) \subseteq \text{tor}(M)$

$$P \oplus N \cong K^{(\mathbb{I})}$$

$$P \rightarrow K^{(\mathbb{I})}$$

$$\varphi(\text{tor}) \subseteq \text{tor}(K^{(\mathbb{I})}) \stackrel{\text{ker je } K^{(\mathbb{I})} \text{ prost}}{=} 0$$

Zaradi injektivnosti φ je tudi $\text{tor}(P) = 0$

$$\begin{array}{c} \text{prost} \xRightarrow{K \text{ cel}} \text{proj.} \xRightarrow{} \text{tarz. prost} \\ \quad \quad \quad \leftarrow \quad \quad \quad \\ \quad \quad \quad K \text{ glavn: } V \\ \quad \quad \quad \text{konohogeneriran} \\ \quad \quad \quad \text{modul} \end{array}$$

Kategorije

Top , Grp , Ab , Vect_F , $K\text{-Mod}$...

$K\text{ mod}$:

objekti: K -moduli

morfizmi: homomorfizmi; K -modulni

$$\text{Hom}_K(P, _) : K\text{ mod} \rightarrow \text{Ab}$$

$$K M \mapsto \text{Hom}(P, M)$$

$$(M \xrightarrow{f} N) \mapsto \text{Hom}(P, M) \rightarrow \text{Hom}(P, N)$$

$$f \mapsto f \circ g$$

3. naloge

$\text{Hom}(P, _)$ je funktor

- ohranja kompozitum
- ohranja identiteto

$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$F(g \circ f) = (f \mapsto f \circ g)$$

$$\begin{aligned} F(g) \circ F(f) (x) &= (f \mapsto f \circ g) (f \mapsto f \circ f) (x) = \\ &= (f \mapsto f \circ g) \circ f = x \circ f \circ g = (f \mapsto f \circ (f \circ g)) = F(f \circ g) \end{aligned}$$

ups, zamenjan vrsta: red

$$F(\text{id}) = (f \mapsto \text{id} \circ f) = \text{id}$$

$P \text{ proj} \Leftrightarrow \underbrace{\text{Hom}(P, -)} \text{ eksakten}$

$$0 \rightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \rightarrow 0$$

$$0 \rightarrow \text{Hom}(P, L) \xrightarrow{F(P)} \text{Hom}(P, M) \xrightarrow{F(N)} \text{Hom}(P, N) \rightarrow 0$$

$\text{Hom}(P, -)$ je **eksakten** če slike eksaktne zaporedja v eksakten

$$\Rightarrow 1) P = K \quad \text{Hom}(K, \underline{M}) \cong_K M$$

$$p \mapsto p \circ f \quad f \mapsto f(1)$$

$$\begin{array}{ccc} \text{Hom}(P, L) & \xrightarrow{F(P)} & \text{Hom}(P, M) \\ \parallel \wr & & \parallel \wr \\ L & \xrightarrow{f} & M \end{array} \quad \begin{array}{c} \downarrow \\ f(1) \end{array}$$

$$2) P \text{ je prost} \quad P \cong K^{(I)}$$

$$(\text{Hom}, -) = \text{Hom}(\bigoplus_I K, -) = \bigoplus_I \text{Hom}(K, -)$$

$$0 \rightarrow \bigoplus \text{Hom}(K, L) \rightarrow \bigoplus \text{Hom}(K, M) \rightarrow \dots$$

$$(l_1, \dots, l_n) \mapsto (f(l_1), f(l_2), \dots)$$

8.12

Y 1.ura

Tenzorski produkt

$\kappa M, \kappa N$ moduli, κ komutativen $\swarrow$ n : nužno, ampak je bolj enostavno če je

$$M \otimes_{\kappa} N = \left\{ \sum_{i=1}^p m_i \otimes n_i \mid m_i \in M, n_i \in N, l \in \mathbb{N}_0 \right\}$$

1) bilinearnost

$$(m+m') \otimes n = m \otimes n + m' \otimes n$$

$$m \otimes (n+n') = m \otimes n + m \otimes n'$$

2) asociativnost $x \in \kappa$

$$x \sum m_i \otimes n_i = \sum x(m_i \otimes n_i) = \sum (xm_i) \otimes n_i = \sum m_i \otimes (xn_i)$$

$$\varphi: \kappa M \times \kappa N \xrightarrow{\text{bilin}} \kappa L$$

$$\begin{array}{ccc} (m, n) & \xrightarrow{\quad} & \varphi(m, n) \\ \downarrow & \searrow & \uparrow \\ m \otimes n & & \varphi(m, n) \end{array} \quad \begin{array}{l} \text{bilinearna} \\ \varphi(xm, n) = x \varphi(m, n) = \varphi(m, xn) \end{array}$$

$\exists! \bar{\varphi}$

U, V vek. prostora $\swarrow$ baza $U = \text{span}\{e_i\} \quad V = \text{span}\{f_j\}$

$$U \otimes V = \text{span}\{e_i \otimes f_j\}$$

1. Zeig:

$$\left(\frac{1}{2}, \frac{1}{2}\right) \otimes (1, 1) + \left(\frac{1}{2}, \frac{1}{2}\right) \otimes (0, 1) + (-1, 0) \otimes \left(\frac{1}{2}, 1\right) \in \mathbb{R}^2 \otimes \mathbb{R}^2$$

ist ein Skalarprodukt (Tensor $u \otimes v$)

$$\begin{aligned} & \left(\frac{1}{2}, \frac{1}{2}\right) \otimes (1, 0) + \left(\frac{1}{2}, \frac{1}{2}\right) \otimes (0, 1) + \left(\frac{1}{2}, \frac{1}{2}\right) \otimes (0, 1) + \frac{1}{2}(-1, 0) \otimes \left(\frac{1}{2}, 1\right) + \\ & \quad (-1, 0) \otimes (0, 1) - \\ &= \left(0, \frac{1}{2}\right) \otimes (1, 0) + (0, 1) \otimes (0, 1) = \\ &= \underline{\underline{\left(0, 1\right) \otimes \left(\frac{1}{2}, 1\right)}} \end{aligned}$$

...

2.

Poisson prime vek. prost. U, V in tenzors

$x \in U \otimes V$, u_i ni enostaven

$$U, V = \mathbb{R}^2$$

$$U', V' = \mathbb{R}$$

$$U', V'$$

$$\sum (x_i \otimes y_i) = \sum x_i y_i (1 \otimes 1) = \left(\sum x_i y_i \right) (1 \otimes 1)$$

~~enostaven~~

$$M \otimes_k K \cong M$$

$$(1, 0) \otimes (1, 0) + (0, 1) \otimes (0, 1) = x \otimes y$$

$$x = (x_1, x_2)$$

$$y = (y_1, y_2)$$

$$x \otimes y = (x_1(1, 0) + x_2(0, 1)) \otimes (y_1(1, 0) + y_2(0, 1)) =$$

$$= x_1 y_1 (1, 0) \otimes (1, 0) + x_2 y_1 (0, 1) \otimes (1, 0) + x_1 y_2 (1, 0) \otimes (0, 1) + x_2 y_2 (0, 1) \otimes (0, 1)$$

$$x_1 y_1 \neq 0 \Rightarrow x_1 y_1 \neq 0$$

$$x_2 y_1 = 0 \Rightarrow x_2 = 0$$

$$x_1 y_2 = 0$$

$$x_2 y_2 \neq 0 \Rightarrow x_2 \neq 0$$

f

15.12

1. wa +

$$L^N = \text{Hom}(N, L)$$

$$\text{Ali: velja } (L^N)^M \cong L^{N \times M}$$

$$3. \text{ Pokazi } {}_K \text{Hom}(M \otimes_K N, L) \cong \text{Hom}(M, \text{Hom}(N, L))$$

$$\begin{aligned} \Phi: {}_K \text{Hom}(M \otimes N, L) &\longrightarrow \text{Hom}(M, \text{Hom}(N, L)) \\ (\varphi: M \otimes N, L) &\longmapsto (\psi: m \mapsto (n \mapsto \varphi(m \otimes n))) \end{aligned}$$

$$\Psi: \text{Hom}(M, \text{Hom}(N, L)) \longrightarrow \text{Hom}(M \otimes N, L)$$

$$\psi \longmapsto \varphi$$

$$(m, n) \mapsto \psi(m)(n)$$

Dokazati moramo še da je bilinearan

$$\psi(m+m')(n) = (\psi(m) + \psi(m'))(n) = \psi(m)(n) + \psi(m')(n)$$

↑
ψ homomorfizem

$$\psi(m)(n+n') = \psi(m)(n) + \psi(m)(n')$$

↑
ψ(m) je homomorfizem

$$\Phi \circ \Psi = \text{id}$$

$$(\Phi \circ \Psi)(\psi)(m)(n)$$

$$\Phi(\psi(m)(n)) = \Phi(m \otimes n) \mapsto \psi(m)(n) = m \mapsto n \mapsto \psi(m, n)(n)$$

$$\psi(m, n)$$

✓ p anemone
rozleg

...

zauważ, iż te relacje dają

$$(\oplus M_i) \otimes N \cong \oplus (M_i \otimes N)$$

V

paralela sem tubo



2025 1. izpit

K cel M, N ta prosti

$M \otimes_K N$ neničen

$$K \otimes_K M = M$$

Razširitev skalarjev

to je produkt dveh vekt. p-št.
zato je produkt
neničen

$$\begin{aligned} (M \otimes_K N) \otimes_K F &= (M \otimes_K F) \otimes_F (N \otimes_K F) \cong (M \otimes_K F) \otimes_F (F \otimes_K N) \\ &\cong (M \otimes_K (F \otimes_F F)) \otimes_K N = (M \otimes_K N) \otimes_K F \end{aligned}$$

$$\begin{aligned} \varphi: M &\rightarrow M \otimes_K F \\ m &\mapsto m \otimes 1 \\ \ker \varphi &= \text{tor } M \end{aligned}$$

ker je M torzijsko prost

ga lahko vložimo v $M \otimes_K F$

$$(M / \text{tor } M) \hookrightarrow M \otimes_K F$$

$$\Rightarrow M \otimes_K F = \{0\} \Rightarrow \ker \varphi = M = \{0\}$$

M kon. gen K -modul, K glavni

tezak: $M \cong K/(d_1) \oplus \dots \oplus K/(d_k) \quad d_1 | d_2 | \dots | d_k$

d_i so endogeno določeni. $\nearrow$ delijo druge

Primer 1) $K = \mathbb{Z} \quad d_i \in \mathbb{Z}$

$$\mathbb{Z}M \cong \mathbb{Z}_{d_1} \oplus \dots \oplus \mathbb{Z}_{d_k} \cong \mathbb{Z}^s \oplus \mathbb{Z}_{d_1} \oplus \dots \oplus \mathbb{Z}_{d_l}$$

2) $K = F[x]$

$$\begin{matrix} \cup \\ F \end{matrix}$$

$$M \cong F^n$$

$$X \cdot m \in F^n \quad \dots \quad X \xrightarrow{n \text{ lin.}} F^n$$

obrat: $A: F^n \xrightarrow{\text{lin.}} F^n \mapsto f(x) \cdot m := f(A)m \in F^n$

$$F^n \cong \frac{F[x]}{f_1(x)} \oplus \dots \oplus \frac{F[x]}{f_n(x)}$$

Danes se mi re da pisat