

DEFINICIJE 1

Navadna diferencialna enačba 1. reda $\dot{x} = F(x, t)$

Skalarne linearne enačbe 1. reda $\dot{x} = f(t)x + g(t)$

Integralne krivulje $\dot{y} = \vec{f}(y(t))$

Ekzistenčni izrek $\cdot x_r(0) = x_0 \quad \cdot \dot{x} = F(t, x)$

$\cdot F$ lipschitzova na $\vec{X} \supset L \quad \cdot F \leq M$ na $[t_0 - a, t_0 + a] \times B_b$

Potem rešitev obstaja in je enoličen na

$[t_0 - \alpha, t_0 + \alpha] \rightarrow B_b$, kjer $\alpha \leq \max \left\{ \frac{1}{2M}, \frac{1}{L} \right\}$

Prvi integral u za $y' = F(x, y)$, če za $\forall$ rešitev $y(x)$

velja $u(x, y(x)) \equiv C$

Pfaffova enačba $P(x, y)dx + Q(x, y)dy = 0$

Integrirajoči množitelj $\lambda(x, y)$, da je $\lambda \cdot F$ potencialno

Lagrange d'Alembertova formula $y = x \cdot \varphi(y') + \psi(y')$

Clairtova enačba $y = x y' + \psi(y')$

Ovojnice družine krivulj $F(x, y, c) \equiv 0$ je $c \mapsto (x(c), y(c))$

za katero velja $\forall c. (x'(c), y'(c)) \perp (\dot{x}(c, t), \dot{y}(c, t))$

DEFINICIJE 2

Picardova preslikava $\mathcal{A}(\vec{x}) = \vec{x}_0 + \int_0^t F(\tau, x(\tau)) d\tau$
Peanov izrek • $F(t, \vec{x}) : [t_0 - a, t_0 + a] \times B_{b, x_0} \longrightarrow \mathbb{R}^n$ zvezna

Potem $\exists x$ da obstaja vsaj ena rešitev $\vec{y} = F(t, \vec{y}), \vec{y}(t_0) = \vec{x}_0$

Tok vektorskega polja $\Phi_t(\vec{x}_0) = \vec{y}(t, x_0)$ za $\vec{y}$ rešitev problema $\dot{\vec{y}} = F(t, \vec{y}) \quad y(t_0) = x_0$

Intenzivizirani: generator $\{\Phi_t\}$ je polje, da velja
 $\left. \frac{d}{dt} \right|_{t=0} \Phi_t(x) = F(x)$

Kompletno polje F , če za $\forall x_0 \in \Omega$ $\vec{x}$ rešitev problema
 $\dot{\vec{x}} = F(\vec{x}), \vec{x}(0) = \vec{x}_0$ definirana za $\forall t \in \mathbb{R}$

Hamiltonski sistem $\Omega \subseteq \mathbb{R}^{2n} \quad \Omega = \{(q_1, \dots, q_n, p_1, \dots, p_n)\}$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Liouvilleova enačba • Φ_t tok vekt. polja F na $\Omega \subseteq \mathbb{R}^{2n}$

$$\frac{d}{dt} \det(J\Phi_t(\vec{a})) = \operatorname{div}(F(\Phi_t(\vec{a}))) \cdot \det(J\Phi_t(\vec{a}))$$

$$\text{Oziram } \det(J\Phi(\vec{a})) = e^{\int \operatorname{div}(F(\tau, \vec{a})) d\tau}$$

Hamiltonsko polje $X_H = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \nabla H$

Nepopolni eliptični integral 1. vrste $F(z, k) = \int_0^z \frac{d\tau}{\sqrt{1-k^2 \sin^2(\tau)}}$

Jacobijeva amplituda $\operatorname{am}(z, k) = F^{-1}(z, k) \nearrow$

Poissonov oklepaj $\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) = X_f \cdot \nabla g$

Liejev produkt vektorskih polj

• antisimetričnost $[F, G] = -[G, F]$

• Jacobijeva identiteta

$$[E, [F, G]] + [G, [E, F]] + [F, [G, E]] = 0$$

Simetrija Hamiltonskega sistema (H, Ω) je difeomorfizem

$S: \Omega \longrightarrow \Omega$, da γ rešitev za (H, Ω) $\gamma(t) = (p(t), q(t))$

$\gamma(0) = x_0$, potem je $S(\gamma(t))$ rešitev za (H, Ω) z

zračnim pogojem $S(\gamma(0)) = S(x_0)$

DEFINICIJE 3

$$\dot{\vec{x}} = A(t) \vec{x}$$

Fundamentalna matrika $\Phi_{t_0, t}$ sistema $\vec{x} = A(t) \vec{x}$ je
 $\Phi = [\vec{x}_1 \dots \vec{x}_n]$ kjer so $\vec{x}_i$ baza rešitev. $\vec{x}_i(t_0) = \vec{e}_i$

Eksponencijacija matrike $e^A = I + \sum_{n=1}^{\infty} \frac{1}{n!} A^n$

BCH formula $e^A e^B = e^C \leadsto C = \ln(e^A e^B)$

Dysonova vrsta $\Phi_t = I + \sum_{n=1}^{\infty} \int_{\sigma_n(t)} A(\tau_1) A(\tau_2) \dots A(\tau_n) d\vec{\tau}$

Liouvillova formula $\det(\Phi_t) = \text{sl}(A(t)) \det(\Phi_t)$
 $\det(\Phi_t) = \det(\Phi_{t_0}) e^{\int_{t_0}^t \text{sl}(A(\tau)) d\tau}$

Determinanta wronskega $w(t) = \det(\Phi_t)$

DEFINICIJE 4

Lagrangeev funkcional $\mathcal{L}(y) = \int_a^b L(x, y(x), y'(x)) dx$

Šibki odvod (smerni, Gateauxov) $P: U \rightarrow V$

$$U_{u_0}^\circ(v) = \lim_{t \rightarrow 0} \frac{1}{t} (P(u_0 + tv) - P(u_0)) = \left. \frac{d}{dt} \right|_{t=0} P(u + tv)$$

Krepi odvod (Frechetov) $P: U \rightarrow V$ $A: U \rightarrow V$ linearna,

omejena. $P(u_0 + v) = P(u_0) + A(v) + o(u_0, v); \lim_{\|v\| \rightarrow 0} \frac{\|o\|}{\|v\|} = 0$

Dopustna funkcija $v \in \{u \in U. u(a) = A, u(b) = B\}$

Dopustna variacija $v \in \{u \in U. u(a) = 0, u(b) = 0\}$

Euler-Lagrangeova enačba $\frac{\partial L}{\partial y}(x, y(x), y'(x)) - \frac{d}{dx} \frac{\partial L}{\partial y'}(\dots) = 0$

Vezani ekstremi: Pogoji $\Phi(u) = l \in \mathbb{R}$, ekstrem $\mathcal{L}$

KAJ MORAM ZNATI

- Kako vemo, da je rešitev sistema globalno definiran
- Dysonova vrsta
- Liouvillova formula
- Vezen: ekstremi
- Euler-Lagrangova enačba
- tangentni prostor je enak jedru vezi
- eksistenčni izrek
- Variacijski račun
- odvod L
- dopustne variacije
- tok in lastnosti
- primeri za eksistenčni izrek
- Fundamentalne metrike

Eksistenčni izrek

• $F(t, \vec{x}) : [t_0 - a, t_0 + a] \times B_b \rightarrow \mathbb{R}^n$ zvezna, Lipschitzova na drugi spremenljivki s konstanto L .

• $\dot{\vec{y}} = F(t, \vec{y}) \quad \vec{y}(t_0) = \vec{x}_0 \in B_{\frac{b}{2}}$

Potem obstaja natanko ena rešitev problema

$$\vec{y} : [t_0 - \alpha, t_0 + \alpha] \rightarrow B_b \quad \alpha \leq \min \left\{ \frac{b}{2M}, \frac{1}{L} \right\}$$

$$M = \max \{ \|F(t, \vec{x})\|; (t, \vec{x}) \in [t_0 - a, t_0 + a] \times B_b \}$$

Dokaz:

$$\vec{x}(t) = \vec{x}_0 + \int_{t_0}^t F(\tau, \vec{x}(\tau)) d\tau$$

$$\mathcal{A}(\vec{x}) = \vec{x}_0 + \int_{t_0}^t F(\tau, \vec{x}(\tau)) d\tau \quad \mathcal{A} : V \rightarrow V$$

$\vec{x}$ je rešitev $\Leftrightarrow \vec{x}(t) = \mathcal{A}(\vec{x}(t)) \Leftrightarrow \vec{x}$ je fiksna točka.

Take točke obstajajo, ko je $\mathcal{A}$ skrajitev

$$P_{a,b} = \{ (t, \vec{x}) \in \mathbb{R} \times \mathbb{R}^n; |t - t_0| < a, \|\vec{x} - \vec{x}_0\| \leq b \}$$

$$E = \{ \vec{y} : \mathcal{I} \rightarrow \mathbb{R}^n; \vec{y}(t_0, \vec{x}) = \vec{x}; \|\vec{y}(t, \vec{x}) - \vec{x}_0\| \leq b \quad \forall (t, \vec{x}) \in P_{a, \frac{b}{2}} \}$$

$$\mathcal{I} = [t_0 - \alpha, t_0 + \alpha] \times B_{\frac{b}{2}}$$

$$\mathcal{A}(E) \subseteq E$$

$$1) \mathcal{A}(\vec{y}(t_0, \vec{x})) = \vec{x} \quad \mathcal{A}(\vec{y}) - \vec{x} = \int_{t_0}^t F(\tau, \vec{y}(\tau, \vec{x})) d\tau = 0$$

$$2) \|\mathcal{A}(\vec{y}(t, \vec{x})) - \vec{x}_0\|_{\infty} \leq \|\vec{x} - \vec{x}_0\| + \int_{t_0}^t \|F(\tau, \vec{y}(\tau, \vec{x}))\| d\tau$$

$$\leq \frac{b}{2} + \alpha M \leq b \quad \text{če} \quad \alpha \leq \frac{b}{2M}$$

Ali je $\mathcal{A}$ na E skrajitev?

$$\|\mathcal{A}(\vec{y}_1) - \mathcal{A}(\vec{y}_2)\| (t, \vec{x}) = \left\| \int_{t_0}^t F(\tau, \vec{y}_1) - F(\tau, \vec{y}_2) d\tau \right\|_{\infty} \leq$$

$$\leq \left\| \int_{t_0}^t L \|\vec{y}_1(\tau, \vec{x}) - \vec{y}_2(\tau, \vec{x})\| d\tau \right\|_{\infty} \leq \alpha L \|\vec{y}_1 - \vec{y}_2\|_{\infty}$$

$$\alpha L < 1 \Rightarrow \alpha < \frac{1}{L}$$

Torej rešitev obstaja

Zvezna odvisnost od začetne točke

$$\vec{y}_1(t, \vec{x}_1) \quad \vec{y}_2(t, \vec{x}_2); \|\vec{x}_1 - \vec{x}_2\| < \delta \quad \vec{y}_1(t, \vec{x}_1) - \vec{y}_2(t, \vec{x}_2)$$

$$\|\mathcal{A}(t, \vec{y}_1(t, \vec{x}_1)) - \mathcal{A}(t, \vec{y}_2(t, \vec{x}_2))\|_{\infty} \leq$$

$$\leq \|\vec{x}_1 - \vec{x}_2\| + \left\| \int_{t_0}^t L \|\vec{y}_1(t, \vec{x}_1) - \vec{y}_2(t, \vec{x}_2)\| dt \right\|_{\infty} \leq$$

$$\leq \delta + \left\| \int_{t_0}^t L \|\vec{y}_1 - \vec{y}_2\|_{\infty} dt \right\|_{\infty} = \delta + L \|\vec{y}_1 - \vec{y}_2\|_{\infty} \cdot \alpha$$

$$\|\vec{y}_1 - \vec{y}_2\|_{\infty} (1 - \alpha L) \leq \delta$$

$$\|\vec{y}_1 - \vec{y}_2\|_{\infty} \leq \frac{\delta}{1 - \alpha L}$$

Tok vektorskega polja

$$\dot{\Phi}_t(x) = F(t, \Phi_t) \quad / \quad \frac{d}{dt}$$

$$(D_x \dot{\Phi}_t)(h) = (D_x F)(t, \Phi_t) \cdot (D_x \Phi_t)(h)$$

$$\leadsto \frac{d}{dt} (D_x \Phi) = D_x F(t, \Phi_t) \cdot D_x \Phi_{t_0} \quad D_x \Phi_{t_0} = I$$

Pomembni lastnosti:

$$1) \Phi_{t_0, t_0}(\vec{x}) = \vec{x}$$

$$2) \Phi_{t, s} \circ \Phi_{s, t_0} = \Phi_{t, t_0}$$

$$\text{Autonomni sistem} \leadsto \Phi_{t, t_0} = \Phi_{t+\tau, t_0+\tau} \Rightarrow \Phi_{t, t_0} = \Phi_{s, 0}$$

$$\leadsto 1) \Phi_0 = I$$

$$2) \Phi_{s+t} = \Phi_s \circ \Phi_t$$

$$\Rightarrow \Phi_t^{-1} = \Phi_{-t} \Rightarrow \Phi_t \text{ je difeomorfizem}$$

Izrek: Liouvilleova enačba

Φ_+ tok avtonomnega polja $F(\vec{x})$ $J\Phi = \det(D\Phi)$

Potem $\frac{d}{dt} \Phi_+(\vec{x}) = \text{div}(F(\Phi_+(\vec{x}))) \Phi_+(\vec{x})$

Dokaz: (za $n=2$)

Oznacimo $\Phi_+ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a(t, x, y) \\ b(t, x, y) \end{bmatrix}$

$$\dot{\Phi}_+ \begin{bmatrix} x \\ y \end{bmatrix} = F(\Phi_+ \begin{bmatrix} x \\ y \end{bmatrix}) \leadsto \begin{bmatrix} \dot{a} \\ \dot{b} \end{bmatrix} = F(\begin{bmatrix} a \\ b \end{bmatrix}) / D_x$$

$$D_{\vec{x}}(\partial_+ \begin{bmatrix} a \\ b \end{bmatrix}) = D_{\vec{x}} \left(\begin{bmatrix} f(a, b) \\ g(a, b) \end{bmatrix} \right)$$

$$\parallel$$

$$(D_{\vec{x}} \begin{bmatrix} a \\ b \end{bmatrix}) = \begin{bmatrix} \partial_x a & \partial_y a \\ \partial_x b & \partial_y b \end{bmatrix}$$

$$\begin{aligned} J &= \partial_+ \det(D_x(\Phi_+ \begin{bmatrix} x \\ y \end{bmatrix})) = \partial_+ \det \begin{bmatrix} \partial_x a & \partial_y a \\ \partial_x b & \partial_y b \end{bmatrix} = \\ &= \partial_+ (\partial_x a \cdot \partial_y b - \partial_y a \cdot \partial_x b) = \\ &= \partial_+ \partial_x a \cdot \partial_y b + \partial_x a \partial_+ \partial_y b - (\partial_+ \partial_y a \cdot \partial_x b + \partial_y a \cdot \partial_+ \partial_x b) = \\ &= \partial_x \partial_+ a \cdot \partial_y b - \partial_y \partial_+ a \cdot \partial_x b + \partial_x a \cdot \partial_y \partial_+ b - \partial_y a \cdot \partial_x \partial_+ b = \\ &= \begin{vmatrix} \partial_x \dot{a} & \partial_y \dot{a} \\ \partial_x b & \partial_y b \end{vmatrix} + \begin{vmatrix} \partial_x a & \partial_y a \\ \partial_x \dot{b} & \partial_y \dot{b} \end{vmatrix} \end{aligned}$$

$$\text{Ker velja } \begin{bmatrix} \dot{a} \\ \dot{b} \end{bmatrix} = F(\begin{bmatrix} a \\ b \end{bmatrix}) = \begin{bmatrix} f(a, b) \\ g(a, b) \end{bmatrix}$$

$$\text{je } \dot{a} = f(a, b) \leadsto \partial_x \dot{a} = f_a \partial_x a + f_b \partial_x b$$

$$\begin{aligned} J &= \begin{vmatrix} f_a \partial_x a + f_b \partial_x b & f_a \partial_y a + f_b \partial_y b \\ \partial_x b & \partial_y b \end{vmatrix} + \\ &+ \begin{vmatrix} \partial_x a & \partial_y a \\ g_a \partial_x a + g_b \partial_x b & g_a \partial_y a + g_b \partial_y b \end{vmatrix} = \end{aligned}$$

$$\begin{aligned} &= f_a (\partial_x a \cdot \partial_y b - \partial_y a \partial_x b) + f_b (\partial_x b \partial_y b - \partial_y b \partial_x b) \\ &+ g_a (\partial_x a \partial_y a - \partial_y a \partial_x a) + g_b (\partial_x a \partial_y b - \partial_y a \partial_x b) = \\ &= \begin{bmatrix} f_a \\ g_b \end{bmatrix} \cdot \begin{vmatrix} \partial_x a & \partial_y a \\ \partial_x b & \partial_y b \end{vmatrix} = \text{div} F(\Phi_+ \begin{bmatrix} x \\ y \end{bmatrix}) \cdot \det(D_x \Phi_+ \begin{bmatrix} x \\ y \end{bmatrix}) \end{aligned}$$

Trditev fundamentalna matrika Φ je obrnljiva. ($\dot{\Phi} = A(t)\Phi$)

Dokaz: Recimo da $\Phi_{t_0, t_1} \vec{a} = \vec{0}$

$$\vec{y}(t) = \Phi_{t_0, t} \vec{a} = \sum_i \vec{a}_i \vec{X}_i(t)$$

$$\vec{y}(t_1) = \vec{0}$$

$\vec{y}$ z začetnim pogojem $\vec{y}(t_1) = 0$

reši enačbo $\dot{y} = F(t, y)$, $y(t_1) = 0$

to reši tudi $z \equiv 0$, torej po eksistencnem

izreku je $y \equiv 0$

torej $\vec{a} \equiv 0$ torej ni netrivialnega jedra.



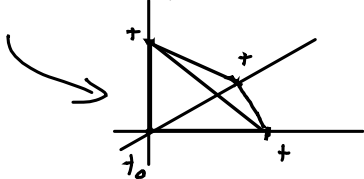
Izrek: Dysonova vrsta

• $A: [t_0, t] \rightarrow \text{Mat}_{n \times n}$ omejena po operatorski normi (norma, kjer $\|I\| = 1$)

Potem je fundamentalna matrika enačbe $\dot{\vec{x}} = A(t)\vec{x}$

$$\Phi_{t, t_0} = I + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} A(\tau_1) A(\tau_2) \dots A(\tau_n) d\tau$$

kjer je $\Delta_n(t) = \{(\tau_1, \dots, \tau_n) \in \mathbb{R}^n; t_0 \leq \tau_1 \leq \dots \leq \tau_n \leq t\}$



Dokaz: Pogledajmo kdaj velja $\dot{\Phi} = A(t)\Phi$ oziroma

$$\Phi = I + \int_{t_0}^t A(\tau) \Phi(\tau) d\tau$$

Rekurzivno kliče enačba sama sebe:

$$\begin{aligned} \Phi &= I + \int_{t_0}^t A(\tau_1) \left(I + \int_{t_0}^{\tau_1} A(\tau_2) \left(I + \int_{t_0}^{\tau_2} A(\tau_3) (I + \dots) \dots d\tau_3 d\tau_2 d\tau_1 \right. \right. \\ &= I + \int_{t_0}^t A(\tau_1) d\tau_1 + \int_{t_0}^t d\tau_1 \int_{t_0}^{\tau_1} A(\tau_2) d\tau_2 + \int_{t_0}^t d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \int_{t_0}^{\tau_2} A(\tau_3) d\tau_3 \dots \\ &= I + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} A(\tau_1) \dots A(\tau_n) d\tau_1 \dots d\tau_n = \text{vrsta} \end{aligned}$$

To je izpeljava formule.

Preverimo ali je res rešitev

$$\begin{aligned} \dot{\Phi}_t &= \partial_t \sum_{n=1}^{\infty} \int_{\Delta_n(t)} \dots = \partial_t \sum_{n=1}^{\infty} \int_{t_0}^t d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \dots A(\tau_1) \dots A(\tau_n) d\tau_n \\ &\quad \partial_t \int_{t_0}^t f(\tau) d\tau = f(t) \\ &= \sum_{n=1}^{\infty} \int_{t_0}^t d\tau_2 \int_{t_0}^{\tau_2} \dots A(t) A(\tau_2) \dots A(\tau_n) d\tau_n = \\ &= A(t) \left(I + \sum_{n=2}^{\infty} \int_{t_0}^t d\tau_2 \int_{t_0}^{\tau_2} \dots A(\tau_2) \dots A(\tau_n) d\tau_n \right) = \\ &= A(t) \left(I + \sum_{n=1}^{\infty} \int_{t_0}^t d\tau_1 \int_{t_0}^{\tau_1} \dots A(\tau_1) \dots A(\tau_n) d\tau_n \right) \\ &\quad \uparrow \\ &\text{preimenujemo spremenljivke in v vrsti predstavimo n-1} \\ &= A(t) \cdot \Phi_t \end{aligned}$$

Preveriti moramo, ali lahko zamenjamo ∂_t in $\sum$ in ali vrsta Φ_t sploh obstaja.

$$\begin{aligned} \|\Phi_t\| &\leq 1 + \sum_{n=1}^{\infty} \int_{t_0}^t d\tau_1 \int_{t_0}^{\tau_1} \dots \|A(\tau_1)\| \|A(\tau_2)\| \dots \|A(\tau_n)\| d\tau_n \leq \\ &\leq 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} M^n = 1 + \sum_{n=1}^{\infty} \frac{(t-t_0)^n}{n!} M^n = \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} ((t-t_0)M)^n = e^{(t-t_0)M} < \infty \end{aligned}$$

Torej Φ_t obstaja.

Ali lahko zamenjamo ∂_t in $\sum$ v dokazu?

$$\sum_{n=1}^{\infty} \int_{\Delta_n(t)} A(\tau_1) \dots A(\tau_n) d\tau \leq \sum_{n=1}^{\infty} \frac{1}{n!} ((t-t_0)M)^n < \infty$$

Torej po Weierstrassovem M-testu je to enakomerna konvergenca, torej $\partial_t \sum \dots = \sum \partial_t \dots$

Izvoli: Formula, ko A_j komutirajo

Če velja $\forall t_1, t_2 \in [0, b]$

$$[A(t_1), A(t_2)] = A(t_1)A(t_2) - A(t_2)A(t_1) = 0$$

Potem za $\dot{\Phi}_+ = A(t)\Phi_+$ $\Phi_0 = I$ velja rešitev

$$\Phi_+ = e^{\int_0^+ A(\tau) d\tau}$$

Dokaz:

$$\Phi_+ = I + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} A(\tau_1) \dots A(\tau_n) d\tau$$

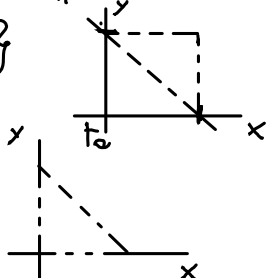
Definirajmo

$$\Delta_n^{\sigma}(t) = \{(\tau_1, \dots, \tau_n) \in \mathbb{R}^n; t_0 < \tau_{\sigma(1)} < \dots < \tau_{\sigma(n)} < t\}$$

za $n=2$ imamo dve permutaciji: $\sigma_1 = id$ $\sigma_2 = (1\ 2)$

$$\Delta_n^{\sigma_1}(t) = \{(x, y); t_0 < x < y < t\}$$

$$\Delta_n^{\sigma_2}(t) = \{(x, y); t_0 < y < x < t\}$$



$$\Delta_2^{\sigma_1}(t) \sqcup \Delta_2^{\sigma_2}(t) = \square_2(t)$$

$$\square_n(t) = \{(\tau_1, \dots, \tau_n) \in \mathbb{R}^n; \forall i, j, i \neq j \Rightarrow \tau_i \neq \tau_j\}$$

↑
brez
diagonala

$$\text{Podobno } \bigsqcup_{\sigma \in S_n} \Delta_n^{\sigma}(t) = \square_n(t)$$

Ker je $\Pi A(\tau_i) = \Pi A(\tau_{\sigma(i)})$ (ker matike komutirajo)

$$\text{velja } \int_{\Delta_n^{\sigma}} A(\tau_{\sigma(1)}) \dots A(\tau_{\sigma(n)}) d\tau = \int_{\Delta_n} A(\tau_1) \dots A(\tau_n) d\tau$$

$$\Rightarrow \int_{\square_n(t)} A(\tau_1) \dots A(\tau_n) d\tau = \sum_{\sigma \in S_n} \int_{\Delta_n^{\sigma}(t)} A(\tau_1) \dots A(\tau_n) d\tau =$$

$$= n! \int_{\Delta_n(t)} A(\tau_1) \dots A(\tau_n) d\tau$$

$$\Rightarrow \Phi_+ = I + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^+ A(\tau_1) d\tau_1 \int_0^+ A(\tau_2) d\tau_2 \dots \int_0^+ A(\tau_n) d\tau_n =$$

$$= I + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\int_0^+ A(\tau) d\tau \right)^n = e^{\int_0^+ A(\tau) d\tau}$$

Izrek: Globalnost rešitev

Naj bo $A(t) \leq M$ za $\forall t \in [t_0, t_1]$

Potem je rešitev $\vec{x}(t)$ za $\dot{\vec{x}} = A(t)\vec{x}$ $\vec{x}(0) = \vec{a}$ definirana na celém intervalu $[t_0, t_1]$

Dokaz:

$\vec{x}(t) = \Phi_{t_0, t} \vec{a}$ Al: Φ_t obstaja povsod?

$$\|\Phi_{t_0, t}\| = \left\| I + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} A(\tau_1) \dots A(\tau_n) d\tau \right\| \leq$$

$$\leq 1 + \sum_{n=1}^{\infty} (t-t_0)^n \cdot \frac{1}{n!} M^n = e^{(t-t_0)M} \leq e^{(t_1-t_0)M} < \infty$$

■

Izrek: Liouvillova formula

$$\text{Naj bo } \dot{\Phi}_t = A(t) \cdot \Phi_t$$

$$\text{Potem velja } \frac{d}{dt} \det(\Phi_t) = \text{sl}(A(t)) \cdot \det \Phi_t$$

$$\text{Dokaz: } \Phi_t = \begin{bmatrix} x_{11}(t) & \dots & x_{1n}(t) \\ \vdots & & \vdots \\ x_{n1}(t) & \dots & x_{nn}(t) \end{bmatrix}$$

$$\det(\Phi_t) = \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} x_{1\sigma(1)} \dots x_{n\sigma(n)}$$

$$\frac{d}{dt} \det(\Phi_t) = \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} \sum_{i=1}^n \dot{x}_{i\sigma(i)} x_{1\sigma(1)} \dots x_{n\sigma(n)} =$$

$$\dot{\Phi} = A(t) \Phi \leadsto \dot{x}_{i\sigma(i)} = \sum_{j=1}^n a_{ij} x_{j\sigma(i)}$$

$$= \sum_{\sigma \in S_n} (-1)^{\text{sgn} \sigma} \cdot \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_{1\sigma(1)} \dots x_{j\sigma(i)} \dots x_{n\sigma(n)} =$$

$$= \sum_{i=1}^n \sum_{j=1}^n a_{ij} \sum_{\sigma \in S_n} (-1)^{\text{sgn} \sigma} x_{1\sigma(1)} \dots x_{j\sigma(i)} \dots x_{n\sigma(n)}$$

imamo σ_1 in $\sigma_2 = (ij) \sigma_1$ za $i \neq j$

$$\begin{aligned} \text{Potem je } & (-1)^{\text{sgn} \sigma_1} x_{1\sigma_1(1)} \dots x_{j\sigma_1(i)} \dots x_{i\sigma_1(j)} \dots x_{n\sigma_1(n)} = \\ & = - (-1)^{\text{sgn} \sigma_2} x_{1\sigma_2(1)} \dots x_{j\sigma_2(j)} \dots x_{i\sigma_2(i)} \dots x_{n\sigma_2(n)} \end{aligned}$$

Torej vsi členi kjer $i \neq j$ se pokrajšajo

$$= \sum_{i=1}^n a_{ii} \sum_{\sigma \in S_n} (-1)^{\text{sgn} \sigma} x_{1\sigma(1)} \dots x_{n\sigma(n)} = \sum_{i=1}^n a_{ii} \det \Phi_t =$$

$$= \text{sl}(A(t)) \det \Phi_t$$

Trditev: A, B diagonalizabilni in $AB=BA$
 Potem obstaja prehodna matrika P , da sta
 PAP^{-1} in PBP^{-1} diagonalni.

Dokaz:

$$Av = \lambda v$$

$$ABv = BAv = B(\lambda v) = \lambda Bv$$

$\Rightarrow Bv$ je lastni vektor za A

$E_\lambda = \{v \in \mathbb{R}^n; Av = \lambda v\}$... prestor lastnih
 vektorjev za lastno vrednost λ

$$v \in E_\lambda \Rightarrow Bv \in E_\lambda$$

Torej B lahko gledamo kot $B_\lambda : E_\lambda \rightarrow E_\lambda$

B_λ lahko diagonaliziramo, ker lahko diagonaliziramo B

$$P_\lambda B_\lambda P_\lambda^{-1} = D_{B_\lambda} \quad \text{v bazi } \{\vec{v}_\lambda^1, \dots, \vec{v}_\lambda^n\}$$

$$Av_\lambda^i = \lambda v_\lambda^i \Rightarrow P_\lambda A P_\lambda^{-1} = \begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix}$$

Tako naredimo za $\forall \lambda$ in dobimo

$$P = \begin{bmatrix} P_1 & & \\ & \ddots & \\ & & P_k \end{bmatrix}$$

da je $PAP^{-1} = D_A$
 in $PBP^{-1} = D_B$

Trditav: če $\mathcal{L}$ v u_0 zavzame lokalni ekstrem,
potem je $\mathcal{D}u_0 \mathcal{L} = 0$

Dokaz:

Recimo da $\forall v \in U. \mathcal{L}(u_0 + tv) > \mathcal{L}(u_0)$

$$\mathcal{L}(u_0 + tv) = \mathcal{L}(u_0) + (\mathcal{D}u_0 \mathcal{L})(tv) + o(u_0, v) > \mathcal{L}(u_0)$$

$$\begin{aligned} (\mathcal{D}u_0 \mathcal{L})(tv) + o(u_0, tv) &> 0 \quad / : t \\ (\mathcal{D}u_0 \mathcal{L})v + \|v\| \frac{o(u_0, tv)}{t\|v\|} &> 0 \quad / \xrightarrow{t \rightarrow 0} \end{aligned}$$

$$(\mathcal{D}u_0 \mathcal{L})(v) + \|v\| \cdot 0 \geq 0 \leadsto$$

$$(\mathcal{D}u_0 \mathcal{L})(v) \geq 0$$

To velja za $\forall v$, tudi za $-v$

$$\Rightarrow -(\mathcal{D}u_0 \mathcal{L})v \geq 0$$

$$(\mathcal{D}u_0 \mathcal{L})v \geq 0 \Rightarrow \mathcal{D}u_0 \mathcal{L} \equiv 0$$



Ena izpeljava

- $\mathcal{L}$ je krepko odvedljiva
- Znamo izračunati žibki odvod

$$(\mathcal{D}_u \mathcal{L})(v) = \frac{d}{ds} \Big|_{s=0} \int_a^b \mathcal{L}(x, u(x) + sv(x), (u'(x) + sv'(x))) dx$$

$$= \int_a^b \left(\frac{d\mathcal{L}}{du} \cdot v(x) + \frac{d\mathcal{L}}{du'} \cdot v'(x) \right) dx$$

$$\int_a^b \frac{d\mathcal{L}}{du'} \frac{dv}{dx} dx = v \cdot \frac{d\mathcal{L}}{du'} \Big|_a^b - \int_a^b \frac{d}{dx} \frac{d\mathcal{L}}{du'} dx$$

$$\Rightarrow (\mathcal{D}_u \mathcal{L})_v = \int_a^b \left(\frac{d\mathcal{L}}{du} - \frac{d}{dx} \frac{d\mathcal{L}}{du'} \right) dx$$

$v(a) = v(b) = 0$

Osnovni izrek variacijskega računa

$u \in \mathcal{C}[a, b]$ poljubna

Če za $\forall p$ testno funkcijo velja $\int_a^b u(x) p(x) dx = 0$,
potem je $u \equiv 0$

Dokaz:

Denimo da $\exists c \in [a, b]. u(c) \neq 0$.

Potem obstaja okolica $[\alpha, \beta]$ za c , da $\forall x \in [\alpha, \beta]. u(x) \neq 0$

Vzemimo testno funkcijo p , da je $\text{supp}(p) = (\alpha, \beta)$

Potem je $\int_a^b u(x) p(x) dx = \int_{\alpha}^{\beta} u(x) p(x) dx \neq 0$

↑
ne zamenj
predznaka

✗
■

Izreki: Euler-Lagrangeova enačba

- U banachov prostor $\mathcal{C}^1[a, b]$ (ima normo)
- $\mathcal{L}: U \rightarrow \mathbb{R} \quad u \mapsto \int_a^b L(x, u(x), u'(x)) dx \quad L \in \mathcal{C}^1$
- $\mathcal{A} = \{u \in U; u(a) = A, u(b) = B\}$
- $\mathcal{L}|_{\mathcal{A}}$ ima v u_0 lokalni ekstrem $\Leftrightarrow u_0$ reši
$$\frac{d}{du} L(x, u(x), u'(x)) - \frac{d}{dx} \frac{d}{du} L(x, u(x), u'(x)) = 0$$

Dokaz:

($\Leftarrow$) sledi iz dejstva $(\mathcal{D}_u \mathcal{L})v = \int_a^b \left(\frac{d}{du} L - \frac{d}{dx} \frac{d}{du} L \right) v(x) dx$

($\Rightarrow$) Recimo da $(\mathcal{D}_u \mathcal{L})v = 0$ za $\forall v$.

Torej velja tudi za v, kjer $v(a) = v(b) = 0$,
to pa so vse testne funkcije.

Torej $\forall \varphi \in$ testna funkcija, $\int_a^b \text{enac}(u_0, x) \cdot \varphi(x) dx = 0$
 $\Rightarrow$ enačba $(u_0, \cdot) \equiv 0$

Vezeni ekstremi

Rešujemo lokalni ekstrem za $\mathcal{L}$ pri pogoju $\Phi(u) \equiv l$

$$V = \{u; \Phi(u) \equiv l\}$$

ekstrem za $\mathcal{L}: V \rightarrow \mathbb{R}$

u bo lokalni ekstrem, če bo vsaka pot v V imela v u lokalni ekstrem za $\mathcal{L}$

$$\gamma: [-\varepsilon, \varepsilon] \rightarrow V$$

Recimo, da ima $\mathcal{L}\gamma(\cdot)$ v 0 lokalni ekstrem

$$u_0 = \gamma$$

$$\forall s \in [-\varepsilon, \varepsilon]. \mathcal{L}\gamma(s) \geq \mathcal{L}\gamma(0) = \mathcal{L}u_0$$

$$\left. \frac{d}{ds} \mathcal{L}\gamma(s) \right|_{s=0} = 0$$

$$\left. \frac{d}{ds} \mathcal{L}\gamma(s) \right|_{s=0} = (D_{\gamma(0)} \mathcal{L}) \left(\overbrace{\left. \frac{d}{ds} \gamma(s) \right|_{s=0}}^v \right)$$

$$= (D_{u_0} \mathcal{L})(v)$$

Tangentni prostor

$$T_u V = \left\{ \underbrace{\frac{d}{ds} \Big|_{s=0} \gamma(s)}_{\gamma'(0)} ; \gamma: [\varepsilon, \varepsilon] \rightarrow V, \gamma(0) = u \right\}$$

Trditelj

$$\bullet \Phi: U \rightarrow \mathbb{R} \text{ vez} \quad \bullet \mathcal{L}: U \rightarrow \mathbb{R}$$

$$\text{Velja } \ker D_u \Phi = T_u V \quad ; V = \{u \in U; \Phi(u) = l\}$$

$$\text{Dokaz: } T_u V \subseteq \ker D_u \Phi$$

$$\Phi(\gamma(s)) = l \quad / \quad \frac{d}{ds} \Big|_{s=0}$$

$$(\mathcal{D}_{\gamma(0)} \Phi) \gamma'(0) = 0$$

$$\Rightarrow T_{\gamma(0)} V \subseteq \ker \mathcal{D}_{\gamma(0)} \Phi$$

$$\ker D_u \Phi \subseteq T_u V$$

$v \in \ker D_u \Phi$ poljubno

Iščemo $\gamma: [\varepsilon, \varepsilon] \rightarrow V$ da je $\gamma'(0) = v$ in $\gamma(0) = u$

Definirajmo $F: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$

$$z, \eta \mapsto \Phi(u + z \cdot v + \eta \cdot w)$$

kjer $v \in \ker D_u \Phi$ in $w \notin \ker D_u \Phi$ $u \in V$

$$F(0,0) = \Phi(u) = l$$

Želimo: bomo izrazili $\eta = \eta(z)$ s pomočjo izreka o

implicitni enačbi: v okolici $(0,0)$

$$\frac{dF}{d\eta} \Big|_{(0,0)} = \frac{dF(0,\eta)}{d\eta} \Big|_{\eta=0} = \frac{d}{d\eta} \Big|_{\eta=0} \Phi(u + \eta w) =$$

$$= (D_{u+\eta w} \Phi) \cdot w \Big|_{\eta=0} = (D_u \Phi) w \neq 0$$

$$\leadsto \eta = \eta(z) \text{ in } \eta(0) = 0$$

Definirajmo $\gamma(s) = u + s \cdot v + \eta(s) \cdot w$

$$\text{Velja } \gamma(0) = u$$

$$\text{želimo še } \gamma'(0) = v$$

$$\gamma'(0) = v + \eta'(0) \cdot w$$

Želimo torej $\eta'(0) = 0$. Preverimo da to drži

$$F(z, \eta(z)) = \Phi(u + z \cdot v + \eta(z) \cdot w) = l \quad / \quad \frac{d}{dz} \Big|_{z=0}$$

$$D_u \Phi(v + \eta'(0)w) = 0$$

$$\underbrace{D_u \Phi v}_{=0} + \underbrace{\eta'(0)(D_u \Phi)w}_{\neq 0} = 0 \Rightarrow \eta'(0) = 0$$

Splošnejša trditev:

- $L: U \rightarrow \mathbb{R}$ funkcional
- $\Phi_i: U \rightarrow \mathbb{R}$ $\alpha_i \in \{1, \dots, n\}$ vezi
- $V = \{u \in U : \forall i, \Phi_i(u) = l_i\}$

Potem $T_u V = \bigcap_{i=1}^n \ker D_u \Phi_i$

$\subseteq$ isto kot prej

$\supseteq$

Naj bo $\{w_1, \dots, w_n\} \subseteq U$ dualna baza za $\{D_u \Phi_1, \dots, D_u \Phi_n\}$. Torej $D_u \Phi_i(u_j) = \delta_{ij}$

Definirajmo: $\gamma_\sigma(s) = u + sv + \sigma_1(s)w_1 + \dots + \sigma_n(s)w_n$
kjer $v \in \ker D_u$.

$$F(s, \vec{\sigma}) = (\Phi_1(\gamma_\sigma(s)), \dots, \Phi_n(\gamma_\sigma(s)))$$

Imamo enačbo $F(s, \vec{\sigma}) = \vec{l}$

Z izrekom o impl.čni preslikavi dobimo $\vec{\sigma} = \vec{\sigma}(s)$
v okolici $(0, 0)$. $F(0, 0) = \vec{l}$ $\sigma(0) = 0$

$$\frac{dF}{ds}(0, 0) = \begin{bmatrix} \frac{dF_1}{ds_1} & \dots & \frac{dF_1}{ds_n} \\ \frac{dF_2}{ds_1} & \dots & \frac{dF_2}{ds_n} \end{bmatrix} (0, 0)$$

$$\left. \frac{dF_i}{ds_j}(0, \vec{\sigma}) \right|_{\vec{\sigma}=0} = \left. \frac{d}{ds_j} \right|_{\vec{\sigma}=0} \Phi_i(u + \sigma_1 w_1 + \dots + \sigma_n w_n) =$$

$$= (D_u \Phi_i) \cdot w_j = \delta_{ij} \Rightarrow D_{(0,0)}^\sigma F = I \Rightarrow \vec{\sigma} = \vec{\sigma}(s)$$

$\Rightarrow \gamma: s \mapsto u + sv + \sigma_1(s)w_1 + \dots + \sigma_n(s)w_n$ je dobro definirano

$$\gamma'(0) = v + \sigma_1'(0)w_1 + \dots + \sigma_n'(0)w_n$$

$$\left. \frac{d}{ds} \right|_{s=0} F(s, \vec{\sigma}(s)) = \begin{bmatrix} \left. \frac{d}{ds} \right|_{s=0} \Phi_1(\gamma(s)) \\ \vdots \\ \left. \frac{d}{ds} \right|_{s=0} \Phi_n(\gamma(s)) \end{bmatrix} =$$

$$= \begin{bmatrix} D_u \Phi_1 \cdot \gamma'(0) \\ \vdots \\ D_u \Phi_n \cdot \gamma'(0) \end{bmatrix} = 0 \Rightarrow \gamma'(0) \in \bigcap_{i=1}^n \ker D_u \Phi_i$$

Lema za splošnejši problem

- $\varphi_1, \dots, \varphi_n$ lin. neodvisni lin. funkcional; $U \rightarrow \mathbb{R}$
- $\bigcap_{i=1}^n \ker \varphi_i \subseteq \ker \varphi$ za nek $\varphi: U \rightarrow \mathbb{R}$

Potem obstajajo λ_i , da velja $\varphi = \sum_{i=1}^n \lambda_i \varphi_i$

Dokaz: Naj bodo $\{v_1, \dots, v_n\} \subseteq U$, da velja

Definirajmo $w = u - \sum_{i=1}^n \varphi_i(u) \cdot v_i$

$$\varphi_k(w) = \varphi_k(u) - \sum_{i=1}^n \varphi_i(u) \cdot \varphi_k(v_i) = \varphi_k(u) - \varphi_k(u) = 0$$

$$\Rightarrow w \in \bigcap_{i=1}^n \ker \varphi_i$$

$$\Rightarrow w \in \ker \varphi$$

$$0 = \varphi(w) = \varphi(u) - \sum_{i=1}^n \varphi_i(u) \cdot \overbrace{\varphi(v_i)}^{\lambda_i}$$

$$\Rightarrow \varphi(u) = \sum_{i=1}^n \lambda_i \varphi_i(u)$$



Splasnjejsi problem

- $\mathcal{L}: U \rightarrow \mathbb{R}$ funkcional
- $\Phi: U \rightarrow \mathbb{R}$ vez
- $u \in U$; $\Phi_i(u) = l_i$ $\mathcal{D}_u \Phi_i \neq 0$

Potem je u lokalni ekstrem $\mathcal{L}$ pri pogojih $\Phi_1 \dots \Phi_n$

$$\Leftrightarrow \exists \lambda_i \in \mathbb{R}. \mathcal{D}_u(\mathcal{L} - \sum \lambda_i \Phi_i)$$

Dokaz: $V = \{u \in U; \forall i. \Phi_i(u) = l_i\}$

Recimo da je u lokalni ekstrem $(\forall v \in V. \mathcal{L}(u) \overset{za v \neq u}{\leq} \mathcal{L}(v))$

Naj bo $\gamma: [\varepsilon, \varepsilon] \rightarrow V$ pot $\gamma(0) = u$

$$\frac{d}{ds} \mathcal{L}(\gamma(s)) = (\mathcal{D}_{\gamma(0)} \mathcal{L}) \gamma'(0) \quad za \quad \forall \gamma$$

$$v \in \ker \mathcal{D}_u \mathcal{L} \Leftrightarrow (\mathcal{D}_u \mathcal{L})v \Leftrightarrow \begin{bmatrix} \mathcal{D}_u \Phi_1 \\ \vdots \\ \mathcal{D}_u \Phi_n \end{bmatrix} v = 0 \Leftrightarrow v \in \bigcap_{i=1}^n \ker \mathcal{D}_u \Phi_i$$

$$\Leftrightarrow \ker \mathcal{D}_u \mathcal{L} = \bigcap_{i=1}^n \ker \mathcal{D}_u \Phi_i$$

$$\Leftrightarrow \mathcal{D}_u \mathcal{L} = \sum_{i=1}^n \mathcal{D}_u \Phi_i \cdot \lambda_i$$

Imamo torej u je ekstrem $\Leftrightarrow \mathcal{D}_u(\mathcal{L} - \sum_{i=1}^n \lambda_i \Phi_i) = 0$

za neke λ_i . Torej ima u ekstrem $\Leftrightarrow$ ko ima

ekstrem $\mathcal{L} - \sum_{i=1}^n \lambda_i \Phi_i$