

(NDE)  
Navadne diferencialna enačba je enačba oblike

$$f(x, y, y') = 0 \quad \text{iščemo } y(x) \text{ da velja}$$

$$f(x, y(x), y'(x)) = 0 \quad \text{za } \forall x \in D_y$$

(To je v implicitni obliki)

Mi se bomo vedno ukvarjali s tem, če je  
podano v eksplisitni obliki, torej ko je  $y' = f(x, y)$

Enačba reda  $n \in \mathbb{N}$  je oblike  $G(x, y, y', \dots, y^{(n)}) = 0$

Enačba je **avtonomna**, če funkcija  $G$  ni  
odvisna od  $x$ . Sicer je neavtonomna.

7.10

1. Za dano družino funkcij poišči  
pripadajočo DE

$$\left. \begin{array}{l} \text{a) } y = ce^x \\ \text{b) } y^2 = cx \\ \text{c) } y = c(x-c) \end{array} \right\} \begin{array}{l} f(x, y, c) = 0 \\ c \in \mathbb{R} \end{array}$$

$$y' = ce^x \Rightarrow y = y'$$

$$y^2 = cx$$

$$2yy' = c \Rightarrow y^2 = 2yy'x$$

$$y = 2y'x \quad \text{za } y \neq 0$$

$$y' = c$$

$$y = y'(x - c)$$

2) ugeni: rešiti naslednjih DE

$$a) y'' = -y \Rightarrow \{ \alpha \cos x + \beta \sin x \}$$

$$b) y + xy' = \cos x$$

$$c) xy' = ny$$

$$b) (xy)' = \cos x$$

$$xy = \sin x + C$$

$$y = \frac{\sin x + C}{x}$$

$$c) xy' = ny$$

$$xy' + y = n \cancel{x} + \cancel{x}$$

$$(xy)' = (n+1)y$$

$$y = x^n$$

# Metoda izoklin

$$y_c = y = y(x, c) \quad x, c \in \mathbb{R}$$
$$y \in \mathbb{C}^1$$

**Izoklina** je krivulja vzdolž katere ima vsaka členica družine  $y$  enak odvod po  $x$  ( $y'_c(x)$ )

$$I_\alpha = \{ (x, y); y = y_c(x) \text{ potem je } y'_c(x) = \alpha$$

če imamo DE oblike  $y' = f(x, y)$  in pripadajočo družino rešitev  $y_c(x)$ , potem so izokline ravno nivojnice  $\{ f = \alpha \}$   
tj.  $y'_c(x) = f(x, y_c(x)) = \alpha$

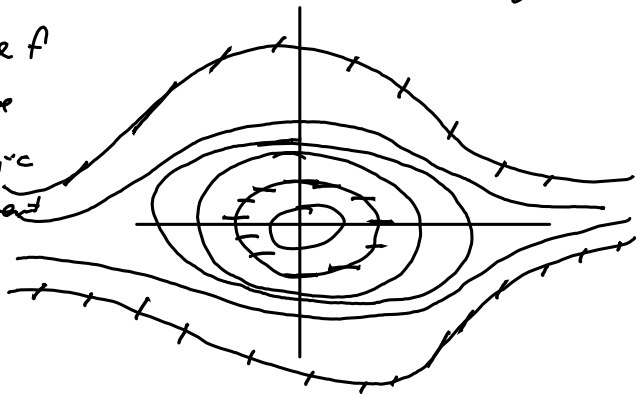
Postopek:

1. skiciramo Runkovo  $f$

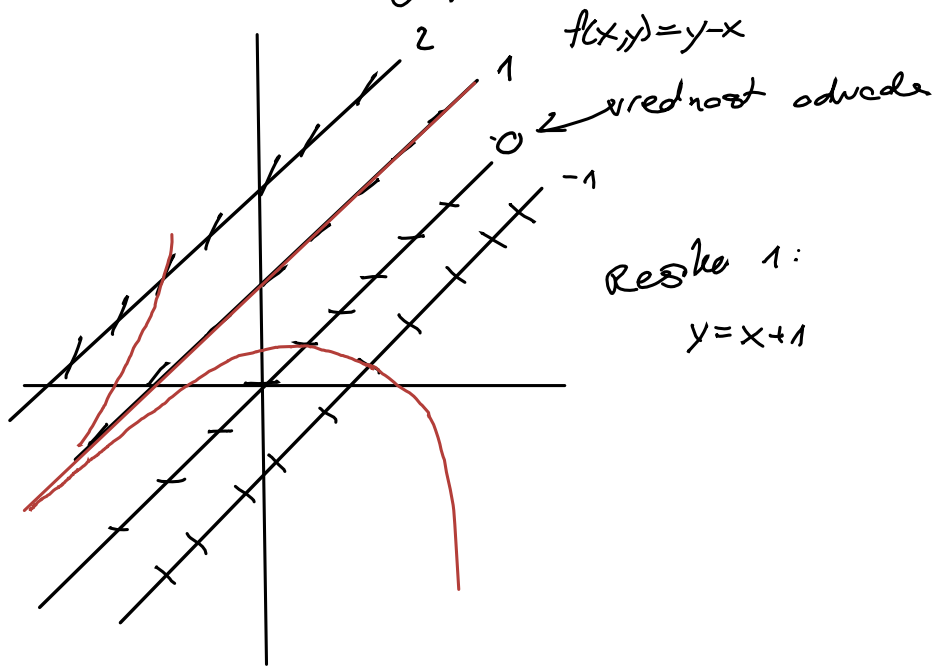
2) vzdolž vsake nivojnice  
nerisamo nekaj delcev  
ki imajo določeno  
enak nivojnici na  
vrednosti  $f$  na  
nivojnici

3) nerisamo krivulj ki so  
v presečiščih nivojnic  
tangente navzgor

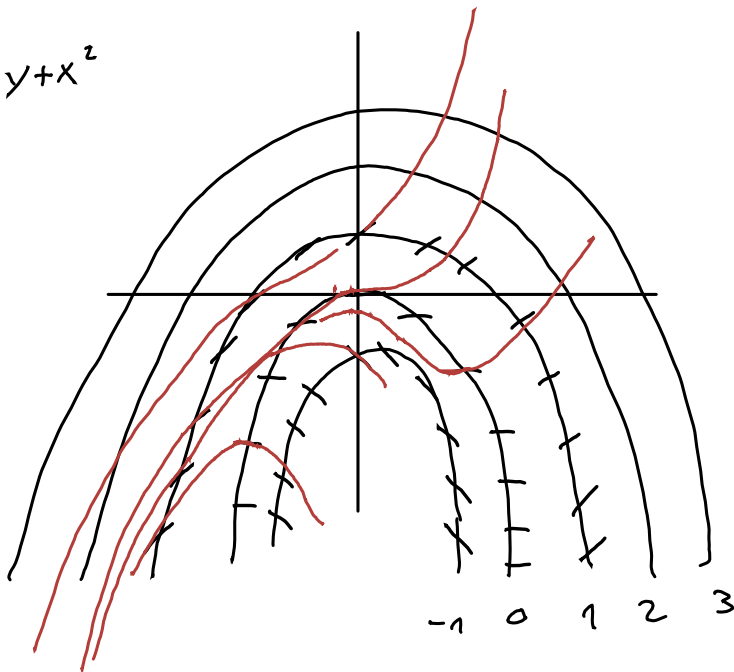
(črna so nivojnice)  
mi iščemo Runkovo



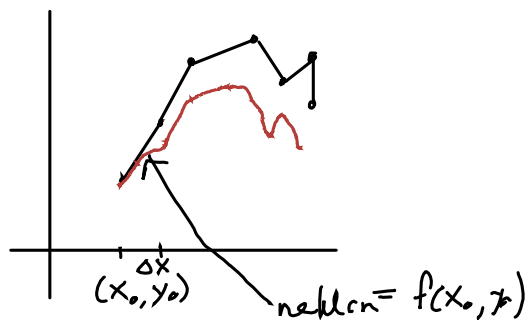
3. Približno skiciraj potek enačbe  $y' = y - x$



$$y' = y + x^2$$



# Eulerjeva metoda



$$x_1 = x_0 + \Delta x$$

$$y_1 = y_0 + \Delta x \cdot f(x_0, y_0)$$

$$x_2 = x_1 + \Delta x$$

$$y_2 = y_1 + \Delta x \cdot f(x_1, y_1)$$

Izberemo si  $\Delta x$

in iterativno  
definiramo  ~~$x_{n+1}$~~

$$x_{n+1} = x_n + \Delta x$$

$$= x_0 + (n+1) \Delta x$$

$$y_{n+1} = y_n + \Delta x \cdot f(x_n, y_n)$$

zakaj imamo smisel?

če privzamemo da je  
rešitev zvezo odvedljiv, in  
velja, da je

$$y(x + \Delta x) \approx y(x) + y'(x) \Delta x + o(\Delta x)$$

5) Z Eulerjevo metodo lokalno rešimo DZ

$y' = f(x)$  kjer je  $f$  zveza pri pogojem  $y(0) = 0$

↙ nekateri  
 $y(A) = ?$

$$\Delta x = \frac{A}{n} \quad \text{za nek } n \in \mathbb{N}$$

$$A = x_n$$

$$x_n = 0 + \Delta x n$$

$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1}) \Delta x \neq y_0 =$$

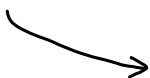
$$= y_{n-1} + f(x_{n-1}) \Delta x + y_0 \quad //^0$$

$$= (f(x_{n-2}) + f(x_{n-1})) \Delta x + y_0$$

$$y_n = \sum_{i=0}^{n-1} f(x_i) \Delta x$$

Riemannova  
 vsota

$$\text{za } \int_0^A f(x) dx$$



$$y_m = \sum_{i=0}^{m-1} f(x_i) \Delta x$$

To je očitno rešitev

Vaja (DN):

Najdi rešitev za  $y' = 2y$  z avterjano metodo

$$\Delta x = \frac{A}{n}$$

$$\begin{aligned} y_n &= y_{n-1} + f(x_{n-1}, y_{n-1}) \Delta x = y_{n-1} + 2y_{n-1} \frac{A}{n} = \\ &= y_{n-1} \left( 1 + \frac{2A}{n} \right) \end{aligned}$$

$$y_n = y_{n-2} \left( 1 + \frac{2A}{n} \right)^2 = y_1 \left( 1 + \frac{2A}{n} \right)^{n-1}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left( 1 + \frac{2A}{n} \right)^{n-1} &= \lim_{u \rightarrow \infty} \frac{\left( 1 + \frac{1}{u} \right)^u}{\left( 1 + \frac{1}{u} \right)} = \underline{\underline{e^{2A}}} \\ \frac{1}{u} &= \frac{2A}{n} \Rightarrow n = 2Au \end{aligned}$$

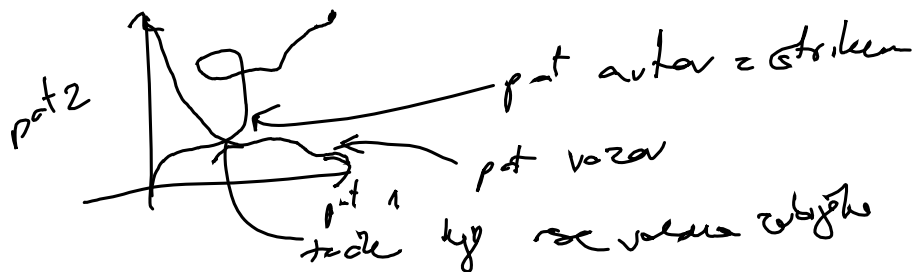
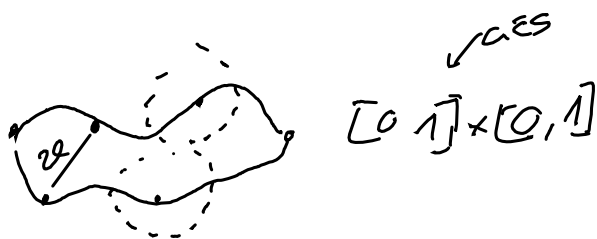
$$y = e^{2x}$$

Fazni prostor je prostor vseh možnih stanj sistema

$$y = f(x, u) \quad f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

↳ fazni prostor je  $\mathbb{R}^3$

- 6) iz Ljubljane v Meribor vodite dve cesti po katerih lahko iz Ljubljane do meribora pripeljete avtomobile ki sta eden na drugega povezane z vrvo dolžine  $< 2l$ , ne da bijo pretrgela. Ali se lahko vozava krožne dolžine radija  $l$ , ki vozita vsek v svojo smer srečata, ne da bi trčila



$$1) \quad a) \quad y' = \frac{x^2}{y} = \frac{dy}{dx}$$

$$x^2 dx = y dy$$

$$\frac{1}{3} x^3 + C = y^2$$

$$y = \pm \sqrt{\frac{1}{3} x^3 + C}$$

b)

$$2x^2 y y' + y^2 = 2$$

$$2x^2 = \frac{2-y^2}{y y'} = \frac{2-y^2}{y \frac{dy}{dx}} = \frac{2-y^2}{y dy} dx$$

$$\frac{dx}{2x^2} = \frac{y dy}{2-y^2}$$

$$u = 2-y^2 \\ du = -2y dy$$

$$-\frac{1}{2} \frac{1}{x} = -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} (\ln u + C)$$

$$\frac{1}{x} = \ln(2-y^2) + C$$

$$C e^{\frac{1}{x}} = 2-y^2$$

$$y = \pm \sqrt{2 - C e^{\frac{1}{x}}}$$

$$c) (1+x^2)y' = y$$

$$\parallel$$

$$(1+x^2) \frac{dy}{dx} = y$$

$$\frac{dy}{y} = \frac{dx}{1+x^2}$$

$$\ln y = \arctan x + C$$

$$y = C e^{\arctan x}$$

$$y \equiv 0$$

$$c') (1+x^3)y' = y$$

$$y_1 \equiv 0$$

$$2) \frac{dy}{y} = \frac{dx}{1+x^3} = \frac{dx}{(x+1)(x^2-x+1)}$$

$$B = -A$$

$$\frac{A}{(x+1)} + \frac{Bx+C}{x^2-x+1}$$

$$x^2: B+A=0$$

$$x: B+C-A=0$$

$$1: A+C=1$$

$$C=1-\frac{1}{3}$$

$$2B+C=0$$

$$\Rightarrow C = -2B = \frac{2}{3}$$

$$3A=1$$

$$\frac{dy}{y} = \frac{1}{3} \frac{dx}{(x+1)} + \frac{-\frac{1}{3}(x-2)}{x^2-x+1} dx$$

$$\int \frac{x-2}{x^2-x+1} dx = \int \frac{1}{(x-\frac{1}{2})^2 + \frac{3}{4}}$$

2)

$$a) y = \tan(2x+3y-1)$$

$$y = \tan z$$

$$z = 2x+3y-1$$

$$z' = 2+3y'$$

$$y' = \frac{z'-2}{3}$$

$$\tan z = \frac{z'-2}{3}$$

$$z' = 3 \tan z + 2$$

$$1. \text{ res'au : } z' = 0 \Rightarrow z = \arctan\left(-\frac{2}{3}\right)$$

$$\Downarrow y' = -\frac{2}{3} \Rightarrow y = \frac{\arctan\left(-\frac{2}{3}\right) - 2x + 1}{3}$$

$$\frac{dz}{dx} = 3 \tan z + 2$$

$$\frac{dz}{3 \tan z + 2} = dx$$

$$x + C = \int \frac{du}{(3u+2)(1-u^2)}$$

$$u = \tan z$$

$$du = \frac{1}{\cos^2 z} dz \Rightarrow dz = \frac{du}{1-u^2}$$

$$\frac{A}{3u+2} + \frac{Bu+C}{1-u^2}$$

$$u^2: B-A=0 \quad A=B$$

$$u: 3C + 2B = 0$$

$$1: 2C+A=1 \Rightarrow A=1-2C$$

$$= \int \left( \frac{-3}{3u+2} + \frac{-3u+2}{1-u^2} \right) du =$$

$$3C+2-4C=0$$

$$C=2 \Rightarrow A=-3=B$$

$$= -\ln\left(u+\frac{2}{3}\right) +$$

$$+ \int \frac{1}{2} \frac{1}{u-1} + \frac{5}{2} \frac{1}{u+1} =$$

$$\frac{A}{1-u} + \frac{B}{1+u} \quad u: A-B=-3$$

$$1: A+B=2$$

$$A=B-3$$

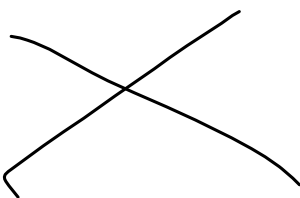
$$2B-3=2$$

$$2B=5$$

$$B=\frac{5}{2}$$

$$A=-\frac{1}{2}$$

$$= -\ln\left(u+\frac{2}{3}\right) + \frac{1}{2} \ln(u-1) + \frac{5}{2} \ln(u+1)$$



$$y''(y-x) = xy'' - (y')^2$$

$$y''(y-x) = y'(1-y')$$

$$y''(y-x) + y'(y'-1) = 0 \quad y=0$$

$$a = y'$$

$$b = y-x \quad b' = y'-1$$

$$a'b + ab' = 0$$

$$(ab)' = 0$$

$$ab = C \in \mathbb{R}$$

$$y'(y-x) = C$$

$$y' = \frac{C}{y-x}$$

$$z = y-x$$

$$z' = y'-1 = \frac{C}{z} - 1$$

~~$$z = \int \left( \frac{C}{z} - 1 \right) dz = C \ln z - z + D$$~~

$$\frac{dz}{dx} = \frac{C}{z} - 1$$

$$\frac{dz}{\frac{C}{z} - 1} = dx \quad x+D = \int \frac{z}{C-z} dz = -\int \left( \frac{C}{t} - 1 \right) dt =$$

$$t = C-z \quad dt = -dz$$

$$= -C \ln(C-z) + C-z$$

$$x+D = -C \ln(C-y+x) + C - y$$

$$y = D e^C e^{\ln(C-y+x)+1}$$

Homogene enade

$$F(x, y) = F(\lambda x, \lambda y) \quad \forall \lambda \neq 0$$

$$\Rightarrow z = \frac{y}{x} \Rightarrow xz = y \Rightarrow y' = z + xz' = F(1, z) \\ z' = \frac{F(1, z) - z}{x} = \frac{f(z)}{g(x)}$$

3)

$$A) y^2 + x^2 y' = xy y'$$

$$z = \frac{y}{x}$$

$$y' (xy - x^2) = y^2$$

$$y' = \frac{y^2}{xy - x^2} = \frac{y^2}{x^2} \cdot \frac{1}{\frac{y}{x} - 1} = \frac{z^2}{z - 1}$$

$$y = zx$$

$$y' = z + z'x = \frac{z^2}{z - 1}$$

$$z'x = \frac{z^2 - z^2 + z}{z - 1} = \frac{z}{z - 1}$$

$$\frac{dz}{dx} x = \frac{z}{z - 1} \Rightarrow \frac{z - 1}{z} dz = \frac{1}{x} dx$$

$$\ln x + D = z + \ln z$$

$$\ln x + D = \frac{y}{x} + \ln \frac{y}{x}$$

$$b) y = xy' - \sqrt{x^2 + y^2}$$

$$y' = \frac{y + \sqrt{x^2 + y^2}}{x} = \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = z + \sqrt{1 + z^2}$$

$$z = \frac{y}{x} \Rightarrow y = zx$$

$$y' = z + z'x = z + \sqrt{1 + z^2}$$

$$\frac{dz}{dx} = \frac{\sqrt{1 + z^2}}{x}$$

$$z = \operatorname{sh} u \quad dz = \operatorname{ch} u du$$

$$\operatorname{arsh} \ln x + D = \int \frac{1}{\sqrt{1 + z^2}} dz = \int \frac{\operatorname{ch} u}{\operatorname{ch} u} du =$$

$$y = x \operatorname{ch}(\operatorname{arsh} \ln x + D)$$

$$= \operatorname{arch} \frac{y}{x}$$

c)

$$ma = F = mg - kv^2$$

$$m\dot{v} = mg - kv^2$$

$$\dot{v} = g - \frac{k}{m}v^2 = g(1 - \frac{k}{mg}v^2) \stackrel{\alpha}{=} g(1 - \alpha v^2)$$

$$\frac{dv}{dt} = g(1 - \alpha v^2)$$

$$\frac{dv}{1 - \alpha v^2} = g dt \quad \int$$

$$gt =$$

$$\frac{A}{1 - \sqrt{\alpha}v} + \frac{B}{1 + \sqrt{\alpha}v} :$$

$$v: \sqrt{\alpha}A - \sqrt{\alpha}B = 0 \\ A = B$$

$$1: A + B = 1$$

$$gt = \frac{1}{\sqrt{\alpha}2} (\ln(1 + \sqrt{\alpha}v) - \ln(\sqrt{\alpha}v - 1)) =$$

$$\Rightarrow A = \frac{1}{2} \\ B = \frac{1}{2}$$

$$\frac{1}{2\sqrt{\alpha}} \ln\left(\frac{1 + \sqrt{\alpha}v}{1 - \sqrt{\alpha}v}\right) = gt + C$$

$$v_0 = 0 \Rightarrow \cancel{1 + \sqrt{\alpha}v} = \cancel{1 - \sqrt{\alpha}v} + C \\ \Rightarrow 0 = C$$

$$\frac{1 + \sqrt{\alpha}v}{1 - \sqrt{\alpha}v} = \underbrace{e^{2\sqrt{\alpha}gt}}_D$$

$$1 + \sqrt{\alpha}v = D - \sqrt{\alpha}vD$$

$$v\sqrt{\alpha}(1 + \sqrt{\alpha}D) = D - 1$$

$$v = \frac{D - 1}{\sqrt{\alpha}(1 + \sqrt{\alpha}D)}$$

V menherje

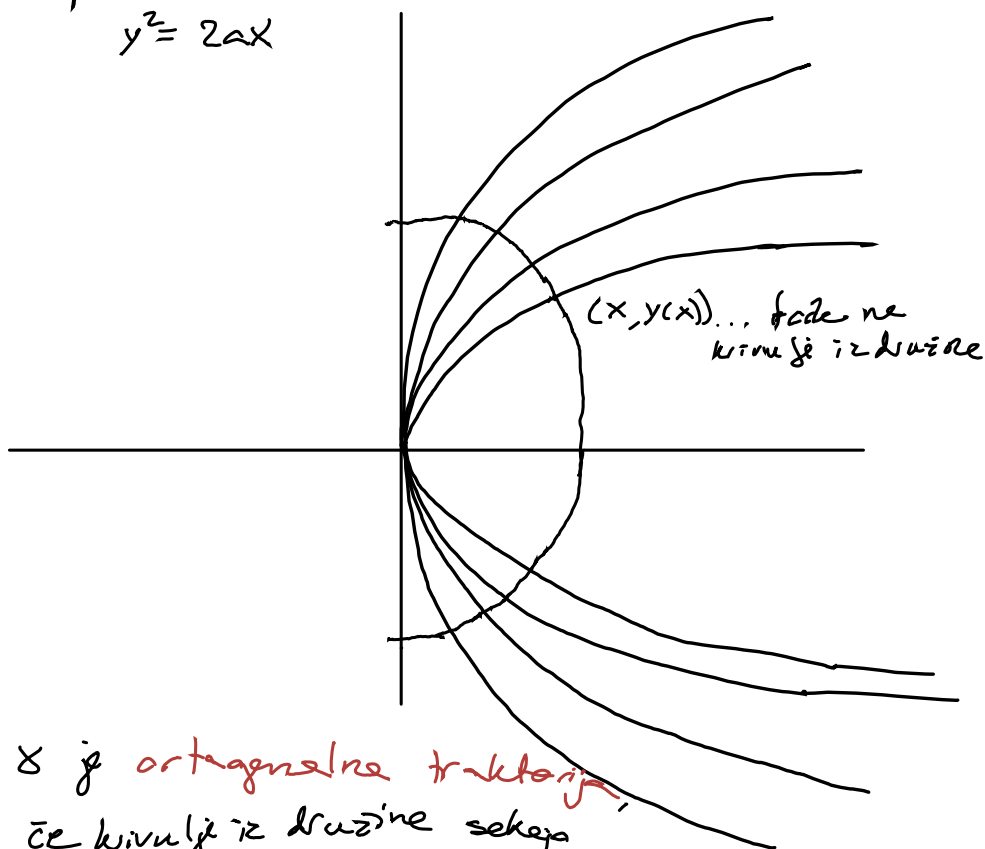
+ zandude 20 min

21.10

2) Poišči družino

ortogonalnih trajektorij na družino  
parabol

$$y^2 = 2ax$$



$\gamma$  je **ortogonalna trajektorija**,

če krivulje iz družine seka

$\gamma$  pod pravim kotom

$$2yy' = 2a \Rightarrow y^2 = 2yy'x$$

$$v(x_0, y_0) \text{ velja } x_0 \cdot 2y_0 y' = y_0^2$$

$$y' = \frac{y_0}{2x_0} \Rightarrow \text{smerni koeficient trajektorije} = -\frac{2x_0}{y_0}$$

$$y' = f(x, y) \text{ družina krivulj} \Rightarrow -\frac{1}{f(x, y)} \text{ družina ort. t.}$$

$$y^2 = 2yy'x \Rightarrow y^2 = -\frac{2yx}{y'} \quad \text{iščemo } y$$

1. možnost  $y=0$  ✓

2. možnost

$$y' = -\frac{2x}{y} = \frac{dy}{dx}$$

$$2x dx = y dy \quad / \int$$

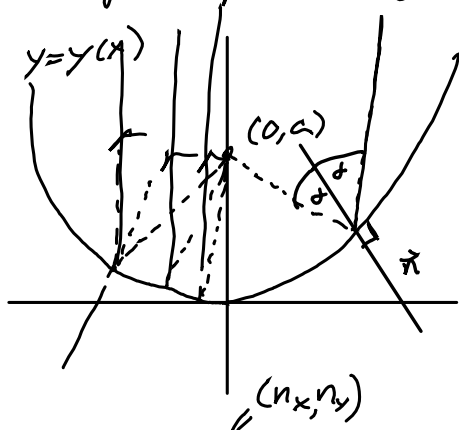
$$-\frac{2}{2} x^2 = \frac{1}{2} y^2 + C$$

$$x^2 + \frac{1}{2} y^2 + C = 0$$

$$x^2 + \frac{1}{2} y^2 = -C \quad \text{elipse}$$

3) Pri izdelavi žarotna mora biti

odbojn površino ki odseva žarotno tako oblike, da vse svetlobne žarke odboji v isto smer predstojimo ležadi, pri čemer rotirajoče ploskev grafa  $f_n$



$$(-x, -y(x)+a) \cdot \vec{n} = -x n_x - y n_y - a n_y = \cos \alpha \cdot \sqrt{x^2 + (y+a)^2} \sqrt{n_x^2 + n_y^2}$$

$$\vec{n} = (1, -\frac{1}{y'})$$

$$\cos \alpha \sqrt{x^2 + (y+a)^2} \sqrt{1 + \frac{1}{y'^2}} = -x + \frac{y-a}{y'}$$

$$\cos \alpha = \frac{-x y' - y - a}{\sqrt{x^2 + (y+a)^2} \sqrt{1 + y'^2}}$$

$$\vec{n} \cdot (1, 0, 0) = \cos \alpha |\vec{n}| \cdot 1$$

$$n_y dx = \cos \alpha \sqrt{1 + \frac{1}{y'^2}}$$

$$n_y \frac{1}{n_x} = \frac{-x y' - y - a}{\sqrt{x^2 + (y+a)^2}} = \frac{1}{y'}$$

$$-x y' + y - a = \sqrt{x^2 + (y+a)^2}$$

???

$$z = y - a$$

$$z' = y'$$

$$x z' + z = \sqrt{x^2 + z^2} + z$$

??

$$z' = \frac{\sqrt{x^2 + z^2}}{x} = \sqrt{\frac{x^2 + z^2}{x^2}} = \sqrt{1 + \frac{z^2}{x^2}} = \sqrt{1 + u^2}$$

$$u = \frac{z}{x}$$

$$z' = x u'$$

$$u + x u' = \sqrt{1 + u^2} + u$$

$$\frac{u'}{\sqrt{1 + u^2}} = \frac{1}{x}$$

$$v = \sqrt{1 + u^2}$$

$$dv = \frac{u du}{\sqrt{1 + u^2}} = \frac{1 - v^2}{\sqrt{1 + u^2}} du$$

$$\ln x = \int \frac{1 - v^2}{v} dv = \ln v - \frac{1}{2} v^2 + C$$

???

$$y = a + \frac{c}{2} x^2 - \frac{1}{2c}$$

# Cauchy nelyže

$$y' = f(y) \quad y(x_0) = y_0$$

- a) gndži.  $\exists$  reši Dž za podpbi zedni poyg  
 b) f lipšicova tolesī kēje resten iz a enliar

$$\frac{dy}{dx} = f(y)$$

$$\frac{dy}{f(y)} = dx \int$$

$$F(y) = x + C$$

$$\int \frac{dy}{f(y)}$$

$$y'(x) = f(y(x))$$

$$y(x) = y(x_0) + \int_{x_0}^x y'(t) dt$$

$$\int_{x_0}^x \frac{y'(t) dt}{f(y(t))} = \int_{x_0}^x 1 dt = x - x_0$$

daš vse na eno stran in integrats

$$u = y$$

$$du = y' dt$$

$$\int_{y_0}^{y(x)} \frac{du}{f(u)} = F(y(x)) - F(y_0) = x - x_0$$

$$F(y(x)) = x - x_0 + F(y_0)$$

$$y(x) = F^{-1}(x - x_0 + F(y_0))$$

$$y(x_0) = F^{-1}(F(y_0)) = y_0$$

$$y'(x) = \frac{1}{F'(F^{-1}(x - x_0 + F(y_0)))} = \frac{1}{f(y(x))}$$

$$= f(y(x)) \quad ???$$

$$F'(F^{-1}(x)) = \frac{1}{(F^{-1})'(x)}$$

b) endlichost

$$\int_{y_0}^y \frac{dy}{f(y)} = x - x_0$$

$$|f(x) - f(y)| \leq C|x - y|$$

$$\begin{array}{c} \nearrow \\ \text{rećmo} \\ \text{da } f(x) = 0 \end{array} \quad |f(y)| \leq C|x - y|$$

$$\int_{y_0}^y \frac{dy}{f(y)} \geq \int_{y_0}^y \frac{dy}{C|y - y_0|} = \infty$$

$f > 0, y > y_0, f(y_0) = 0$

Rećmo da y reći DE da be

$f(y_n) = 0$  in y ni konstanten

$$x_n > x_0 : f(y(x_n)) > 0$$

paćm je y blizu  $x_n$  za koje

$$\int_{x_1}^x \frac{y'(x) dx}{f(y(x))} = x - x_1$$

$\parallel$   
 $\int_{y_1}^y \frac{dy}{f(y)}$

izrećmo si  $x_2 \in [x_0, x_1]$ , da valje

$f(y(x_2)) = 0$  in  $f(y(x)) > 0$  za  $\forall x \in (x_2, x_1]$

$$\lim_{x \rightarrow x_2} \int_{x_1}^x \frac{y'(x)}{f(y(x))} dx = \lim_{x \rightarrow x_2} (x - x_1) = x_2 - x_1$$

$$= \lim_{y \rightarrow y_2} \int_{y_1}^y \frac{dy}{f(y)} \geq \lim_{y \rightarrow y_2} \int_{y_1}^y \frac{dy}{C(y - y_0)} = \infty$$

X

$$y' = \frac{f(y)}{g(x)} \quad f, g \text{ brez ničel}$$

a) Pokušajmo de vsake rešitve de izena krivulje  
polja:  $v(x, y) = (g(x), f(y))$

b) Dokaži za poljken rešitvi posoj točko  
veliku eno rešitve

**Tokovnice** ali **integralna krivulje** je krivulja,  
ki ima vsaki točki odredeno vektorje  
polja

$$F(y) - F(y_0) = \int_{y_0}^y \frac{dy}{f(y)} = t - t_0 \quad x = G^{-1}(t - t_0 + G(x_0))$$

$$G(x) - G(x_0) = \int_{x_0}^x \frac{dx}{g(x)} = t - t_0 \Rightarrow$$

$$\gamma = (x(t), y(t))$$

$$\dot{\gamma}(t) = v(x(t), y(t))$$

$$\dot{x}(t) = g(x)$$

$$\dot{y}(t) = f(y)$$

$$\frac{d}{dt} y(x(t)) =$$

$$= \frac{\partial y}{\partial t}(x(t)) \dot{x} =$$

$$= \frac{f(y(t))}{g(x(t))} \frac{dx}{dt}$$

$$\Rightarrow \dot{x} = g(x) \quad ???$$

✓

## lin. dif. enačbe (LDE)

$$y' + a(x)y = b(x) \quad a, b \in \mathcal{C}([a, b])$$

i)  $b \equiv 0 \Rightarrow$  LDE homogena  $\Rightarrow$  ločljive spremenljivke

$$y = y_0 e^{-\int a(t) dt} = C e^{-\int a dx}$$

ii)  $b \not\equiv 0 \Rightarrow$  LDE nehomogena  $\Rightarrow$  variacija konstante

$$y = e^{-\int a dx} \int b(x) e^{\int a dx} dx + C e^{-\int a dx}$$

•) LDE homogena:  $y_1, y_2$  rešitvi  $\Rightarrow y_1 + \lambda y_2$  tudi rešitev

če vemo homogeno rešitev  $\bar{y}$  so vse druge rešitve oblike  $C \cdot \bar{y}$  za  $C \in \mathbb{R}$

••) LDE nehomogena

Partikularno rešitev najdemo z variacijo konstante

Homogeni del:  $y' + ay = 0 \Rightarrow y_h$  rešitev

Partikularni del:  $y_p = C(y) y_h$

Splasnena rešitev:  $\tilde{y} = y_p + D y_h$

Odvajamo in vstavimo v DE:

$$y_p' = C' y_h + C y_h' \Rightarrow C' y_h + C \tilde{y}_h' + a C y_h = b$$

$$C' y_h = b \Rightarrow C(x) = \int \frac{b}{y_h} dx$$

1) Poisťte sa zbližovať korigu:

$$a) xy' + 2y = (3x+2)e^{3x} \quad / :x$$

$$y' + y \frac{2}{x} = \underbrace{\frac{3x+2}{x}}_b e^{3x}$$

$$x=0 \Rightarrow 2y=2 \Rightarrow y=1$$

$$x=0 \Rightarrow$$

$$\text{homogeno: } y' + y \frac{2}{x} = 0$$

$$\frac{dy}{dx} + y \frac{2}{x} = 0 \quad / \frac{dx}{y}$$

$$\frac{dy}{y} = -\frac{2dx}{x} \quad / \int$$

$$\ln y = \ln x^{-2} + C$$

$$y = \frac{1}{x^2} \cdot D$$

partikularno:

$$y_p = D(x) x^{-2}$$

$$y'_p = D'(x) x^{-2} - 2D(x) x^{-3}$$

$$D'(x) x^{-2} - \cancel{2D(x) x^{-3}} + \cancel{2D(x) x^{-3}} = \frac{(3x+2)e^{3x}}{x}$$

$$D'(x) = (3x^2 + 2x) e^x$$

$$\int p(x) e^{\lambda x} dx = q(x) e^{\lambda x} + C$$

$$p, q \in \mathbb{R}[x], \quad \text{st } p = \text{st } q$$

$$\left( (Ax^2 + Bx + C) e^{3x} \right)' = (3x^2 + 2x) e^{3x}$$

$$(3Ax^2 + 3Bx + 3C + 2Ax + B) e^{3x} \leadsto$$

$$3A = 3 \quad \Rightarrow A = 1$$

$$3B + 2A = 2 \quad \Rightarrow B = 0$$

$$3C + B = 0 \quad \Rightarrow C = 0$$

$$D(x) = x^2 e^{3x}$$

$$\leadsto y_p = x^2 e^{3x} \cdot x^{-2} = e^{3x}$$

$$y_s = y_h + y_p = D \frac{1}{x^2} + e^{3x}$$

$$b) \quad y'' = \sin x + y'$$

$$z = y'$$

$$a(x) = -1 \quad b(x) = \sin x$$

$$z' = \sin x + z$$

$$b \equiv 0 \Rightarrow z' - z = 0$$

$$z' - z = \sin x$$

$$z' = z$$

$$1) \quad z \equiv 0 \Rightarrow y = C \in \mathbb{R}$$

$$2) \quad \int \frac{dz}{z} = dx \Rightarrow \ln z = x + C$$

$$\Rightarrow z = Ce^x \Rightarrow y = Ce^x + D$$

$$b \neq 0 \Rightarrow z_h = e^x$$

$$z_p = C(x)z_h = C(x)e^x$$

$$z = z_p + D y_h = C(x)e^x + D e^x$$

$$z' = C'(x)e^x + C(x)e^x + D e^x$$

$$z' - z = C'(x)e^x + C(x)(e^x - e^x) = \sin x$$

$$C'(x) = \frac{\sin x}{e^x} = e^{-x} \sin x$$

$$\text{nastavek: } C(x) = A e^{-x} \sin x + B e^{-x} \cos x$$

$$C(x) = e^{-x}(A \sin x + B \cos x)$$

$$C'(x) = -e^{-x}(A \sin x + B \cos x) + e^{-x}(A \cos x - B \sin x)$$

$$= e^{-x}(\sin x(-A - B) + \cos x(-B + A)) = e^{-x} \sin x$$

$$A + B = -1$$

$$A - B = 0 \Rightarrow A = B \Rightarrow 2A = -1 \Rightarrow A = B = -\frac{1}{2}$$

$$\Rightarrow C(x) = -\frac{1}{2} e^{-x}(\sin x + \cos x) =$$

$$z = -\frac{1}{2}(\sin x + \cos x) + D e^x = y'$$

$$y = -\frac{1}{2} \int (\sin x + \cos x) dx = \frac{1}{2} \cos x - \frac{1}{2} \sin x + D e^x + C$$

$$2) \quad y' + 2xy = 2x^3 y^3$$

1)  $y \equiv 0$  je rešitev

$$2) \quad y \neq 0 \Rightarrow \frac{y'}{y^3} + \frac{2x}{y^2} = 2x^3$$

$$z' = \frac{y'}{y^3} \Rightarrow \int dz = \int y^{-3} dy \Rightarrow z = -\frac{1}{2}y^{-2} = -\frac{1}{2y^2}$$

Bernullijeva DE

$$y' + ay = by^\alpha \quad \alpha \notin \{0, 1\}$$

$$\text{Nastavek: } z = y^{1-\alpha}$$

$$\frac{2x}{y^2} = \frac{4x}{2y^2} = -4xz$$

$$\text{enačba: } z' - 4xz = 2x^3 \quad a(x) = -4x \quad b(x) = 2x^3$$

$$\text{homogeno: } z' - 4xz = 0$$

$$\int \frac{dz}{z} = \int 4x dx \Rightarrow \ln z = 2x^2 + C \Rightarrow z = Ce^{2x^2}$$

$$\text{partikularno: } z_h = e^{2x^2} \quad z_p = C(x)e^{2x^2}$$

$$z = z_p + z_h$$

$$\leadsto \text{enačba: } C'(x)e^{2x^2} + 4xe^{2x^2} \cdot C - 4xCe^{2x^2} = 2x^3$$

$$C'(x) = e^{-2x^2} \cdot 2x^3$$

Nastavek:

$$C(x) = \int 2x^3 e^{-2x^2} dx = \int x e^u dx = (Ax + B)e^x$$

$$u = -2x^2$$

$$du = -4x dx$$

$$= -\frac{1}{2} \int e^u u du = \frac{1}{2} (Au + B)e^u$$

$$((Au + B)e^u)' = e^u (Au + B + A)$$

$$A + B = 0 \quad A = 1 \Rightarrow B = -1$$

$$C(x) = -\frac{1}{2} (1u - 1)e^u \leadsto$$

$$C(x) = e^{-2x^2} (x^2 - \frac{1}{2}) + D$$

$$z = x^2 - \frac{1}{2} + De^{2x^2} \leadsto y = \frac{1}{\sqrt{x^2 - \frac{1}{2} + De^{2x^2}}}$$

### 3. nelože

$$xy' + y = y^2 \ln x$$

1)  $y = 0 \Rightarrow$  je rešitev

2)  $y \neq 0 \Rightarrow$   
 $x \neq 0 \Rightarrow$

$$\frac{y'}{y^2} + \frac{1}{xy} = \ln x$$

$$z' = \frac{y'}{y^2} \leadsto z = -\frac{1}{y}$$

$$\boxed{z' - \frac{z}{x} = \ln x} \quad \begin{matrix} a = -1 \\ b = \ln x \end{matrix}$$

homogeno:

$$z' - \frac{z}{x} = 0$$

$$\frac{dz}{z} = \frac{dx}{x} \leadsto$$

$$\ln z = \ln x + C$$

$$z = e^{\ln x} \cdot e^C \leadsto z = Cx$$

$$z_h = x$$

$$\leadsto C'x + C - C = \ln x$$

$$C' = \frac{\ln x}{x} \quad C = \int \frac{\ln x}{x} dx = \int u du = \frac{1}{2} u^2 + D$$

$$\begin{matrix} u = \ln x \\ du = \frac{dx}{x} \end{matrix} \quad C = \frac{1}{2} (\ln x)^2 + D$$

$$z = \frac{1}{2} (\ln x)^2 x + Dx = -\frac{1}{y}$$

$$y = -\frac{2}{(\ln x)^2 x + Dx}$$

4 naloga

$f$  omejena zvezna

$$y' + y = f(x)$$

a) To ima natanko eno rešitev DE

b)  $f$  periodična s periodo  $w \Rightarrow \bar{y}$  periodična s periodo  $w$

a)  $y' + y = f(x)$

homogeno:

$$\frac{dy}{dx} = -y \leadsto \frac{dy}{y} = -dx \leadsto y = e^{-x}$$

partikularna:  $y_p = C(x)e^{-x}$

$$C'(x)e^{-x} = f(x) \quad C(x) = \int_0^x f(t)e^t dt = F(x) + D$$

$$\bar{y} = F(x)e^{-x} + Ce^{-x} = e^{-x}(F(x) + C) = e^{-x} \int_0^x f(t)e^t dt$$

b) 
$$\begin{aligned} \bar{y}(x+w) &= e^{-x-w} \int_0^{x+w} f(t)e^t dt = e^{-x}e^{-w} \int_0^x f(t)e^t dt + \\ &\quad e^{-w}e^{-x} \int_x^{x+w} f(t)e^t dt = \\ &= e^{-w}\bar{y} + e^{-w}e^{-x} \int_x^{x+w} f(t)e^t dt = e^{-w}e^{-x} \int_0^x f(t)e^t dt + e^{-w}e^{-x} \int_x^{x+w} f(t)e^t dt = \\ &= e^{-w}\bar{y}(x) + e^{-x}\bar{y}(w) - e^{-w}\bar{y}(x) = e^{-x}\bar{y}(w) = e^{-w}\bar{y}(x) \end{aligned}$$

$$\begin{aligned} \bar{y}(x+w) &= e^{-x-w} \int_{-\infty}^{x+w} f(t)e^t dt = e^{-x-w} \int_{-\infty}^w f(t)e^t dt + e^{-x-w} \int_w^{x+w} f(t)e^t dt = \\ &= e^{-x-w} \left( \int_{-\infty}^w f(t)e^t dt + \int_w^x f(t)e^t dt \right) = \\ &= e^{-x}\bar{y}(w) + e^{-w}\bar{y}(x) \end{aligned}$$

???

$$b) \bar{y}(x) e^{-x} \int_{-\infty}^x f(t) e^t dt = \int_{-\infty}^x f(t) e^{t-x} dt = \int_{-\infty}^0 f(u+x) e^u du$$

$$u = t-x$$

$$\Rightarrow t = u+x$$

$$\bar{y}(w+x) = \int_{-\infty}^0 f(u+x+w) e^u du = \int_{-\infty}^0 f(u+x) e^u du = \bar{y}(x)$$

5 nelinear

$$3y' + y^2 + \frac{2}{x^2} = 0$$

Ugnetimo rešitev:  $y = \frac{1}{x}$   $y' = -\frac{1}{x^2}$

$$-\frac{3}{x^2} + \frac{1}{x^2} + \frac{2}{x^2} = 0 \quad \checkmark$$

Splāšna rešitev:  $y = \frac{1}{x} + u(x)$   
 $y' = -\frac{1}{x^2} + u'(x)$

$$-\frac{3}{x^2} + 3u'(x) + \frac{1}{x^2} + \frac{2u(x)}{x} + u^2(x) + \frac{2}{x^2} = 0$$

$$3u'(x) + \frac{2u(x)}{x} + u^2(x) = 0 \quad / : u^2(x)$$

$$\frac{3u'(x)}{u^2(x)} + \frac{2}{xu(x)} + 1 = 0$$

$$z' = \frac{3u'(x)}{u^2(x)} \quad z = 3 \int u^{-2} du = -3u^{-1} = -\frac{3}{u}$$

$$\Rightarrow u = -\frac{3}{z}$$

$$z' - \frac{2}{3} \frac{z}{x} = -1$$

$$a(x) = \frac{2}{3x}$$

$$b(x) = -1$$

homogeni del:

$$\frac{dz}{z} = \frac{2}{3} \frac{dx}{x}$$

$$\ln z = \ln x^{\frac{2}{3}} + C \Rightarrow z = \frac{2}{3} Cx$$

partikulārs:

$$C'(x) \frac{2}{3} x = -1$$

$$C(x) = -\frac{3}{2} \int \frac{1}{x} dx = \ln x^{-\frac{3}{2}}$$

$$z = x \ln x + \frac{2}{3} Cx$$

$$u = -\frac{3}{x \ln x + \frac{2}{3} Cx} \Rightarrow y = \frac{1}{x} - \frac{3}{x \ln x + \frac{2}{3} Cx}$$

## Ricattijeva enačba

$$y' = ay^2 + by + c \quad a, b, c \text{ funkcije } x$$

1) Uagremo rešitev  $y_p$

2) Splošno rešitev poiščemo z nastavekom

$$y = y_p + u(x)$$

3) Vstavimo nastavek z začetno enačbo

4) Dobimo Bernullijevo enačbo

$$z) \quad y = y_p + \frac{1}{u^2(x)} \Rightarrow LDE$$

1.  $(1-x^2)y' = 1-y^2$

Riccatijeva enačba

$$y' = ay^2 + by + c \quad ; \quad a, b, c \text{ func. od } x$$

1) Uganimo rešitev  $y_p$

2) splošna rešitev kot  $y = y_p + u(x)$

3) Dobimo Bernolijeva enačba

$$y = y_p + \frac{1}{u(x)^2} \Rightarrow \text{CDE}$$

$$y' = \frac{1-y^2}{1-x^2}$$

1.  $y_p = x : 1 = \frac{1-x^2}{1-x^2} \quad \checkmark$

2.  $y = v + y_p = v + x$   $v$  je neznan funkcija

$$y' = v' + 1 = \frac{1 + (v+1)^2}{1-x^2} = \dots = 1 + \frac{-2vx - v^2}{1-x^2}$$

$$v' = \frac{-2vx - v^2}{1-x^2} = -\left(\frac{1}{1-x^2} v^2 + \frac{2x}{1-x^2} v\right)$$

$$\frac{v'}{v^2} = -\frac{1}{1-x^2} - \frac{2x}{1-x^2} \frac{1}{v}$$

$$z = \frac{1}{v} \quad z' = \frac{v'}{v^2}$$

$$z' = -\frac{1}{1-x^2} - \frac{2x}{1-x^2} z$$

homogeni del:

$$z' = -\frac{2x}{1-x^2} z$$

$$\int \frac{dz}{z} = -\int \frac{2x}{1-x^2} dx$$

$$\ln|z| = -\ln|1-x^2| + C$$

$$z_h = \frac{C}{x^2-1}$$

$$\frac{C'(x)}{x^2-1} + \frac{C}{x^2-1} = \dots = -\frac{1}{x^2-1}$$

$$C'(x) = -1 \quad C = -x$$

$$z = z_h + z_p = \frac{C-x}{x^2-1}$$

$$v = \frac{x^2-1}{C-x}$$

$$y = \dots = \frac{1-Cx}{x-C}$$

Prvi integral enačbe  $y' = f(x, y)$  je funkcija, ki je konstantna vzdolž vsake rešitve

Če imamo splošno rešitev  $y = \varphi(x, C)$  in je  $\varphi'_C \neq 0$  ↓ odvod po C  
 $\Rightarrow C = C(x, y)$  je prvi integral

1) DE zapišemo v obliki  $Pdx + Qdy = 0$  (\*)

Opomba: (\*) je ekvivalentna DE  $P + Qy' = 0$

$$\langle (P, Q), (1, y') \rangle = 0$$



če velja  $Q_x = \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = P_y$

preverimo da je enačba ~~eksaktna~~ eksaktna.  
 Njen prvi (eksaktni integral) poiščemo  
 z integracijo  $u = \int$   
 $\hookrightarrow (P, Q)$  je potencialno  $\Leftrightarrow (P, Q) = \nabla u$

C poiščemo z odvajanjem

$$u_y = \frac{\partial}{\partial y} \int P dx + C'(y) = Q$$

2) Poišči kak nekonstanten prvi integral

$$a) x\sqrt{x^2+y^2}+y)dx + (\sqrt{x^2+y^2}+x)dy = 0$$

$$b) 3x^2(1+\ln y)dx - (2y - \frac{x^3}{y})dy = 0$$

a)

$$P = x\sqrt{x^2+y^2}+y$$

$$Q = y\sqrt{x^2+y^2}+x$$

Ali je eksaktna?

$$P_x = \sqrt{x^2+y^2} + \frac{xy}{\sqrt{x^2+y^2}}$$

$$Q_y = -1$$

$$u = \int (x\sqrt{x^2+y^2} + \frac{y}{\sqrt{x^2+y^2}}) dx + C(y) =$$

$$t = x^2+y^2$$

$$dt = 2x dx$$

$$= \frac{1}{2} \int \frac{1}{t^{\frac{1}{2}}} dt + xy + C(y) = \frac{1}{3} (x^2+y^2)^{\frac{3}{2}} + xy + C(y)$$

$$u_y = \frac{y(x^2+y^2)^{\frac{1}{2}}}{1} + C'(y) = y\sqrt{x^2+y^2} + C'(y)$$

$$C'(y) = 0 \Rightarrow C \text{ je konstanta}$$

$$u = \frac{1}{3} (x^2+y^2)^{\frac{3}{2}} + xy \quad \text{preverimo da je 0}$$

↑ prvi integral

splošna rešitev dobimo če iz  $u(x,y) = C$

izrazimo  $y$   $\tilde{y} = \tilde{y}(x)$

$$u(x, \tilde{y}(x)) = C$$

1

Da je  $Pdx + Qdy = 0$  (\*) moč veljati

$$y' = -\frac{P}{Q} = -\frac{\mu P}{\mu Q} \Rightarrow \mu Pdx + \mu Qdy = 0 \quad (**)$$

Ti dve sta ekvivalentni za poljubno rešitev  
če \* ni eksakten mora biti \*\* eksakten  
faktor  $\mu$  se imenuje integracijski množilec

V praksi ga skušamo ugotoviti z nastevki:

$$\mu(x), \mu(y), \mu(x, y)$$

$$\mu(x^n \pm y^m), \mu(x^2 + y^2) \dots$$

3) Podana je DE

$$Pdx + Qdy = 0$$

a) Dokazi  $\mu = \mu(y) \Leftrightarrow \frac{Q_x - P_y}{P} = g(y)$

$\Rightarrow \mu Pdx + \mu Qdy = 0$  in  $\mu$  integrirajoči množitelj  
DE pa je eksaktna

$$(\mu P)_y = (\mu Q)_x$$

$$\mu_y P + \mu P_y = \mu Q_x$$

$$\frac{\mu(Q_x - P_y)}{P} = \mu'(y) \quad /: \mu$$

$$\frac{Q_x - P_y}{P} = \frac{\mu'(y)}{\mu(y)} \quad \int \text{odvisno samo od } y$$

$$\Leftrightarrow \frac{Q_x - P_y}{P} = g(y) \quad \text{recimo da je}$$

$$\ln \mu = \int \frac{Q_x - P_y}{P} dy \Rightarrow \mu = e^{\int \frac{Q_x - P_y}{P} dy}$$

Potem je  $(\mu P)_y = \frac{Q_x - P_y}{P} P + \dots$  itak itak  
pride vredn

$$b) \underbrace{(xy^2 - y^3)}_P dx + \underbrace{(1 - xy^2)}_Q dy = 0$$

$$Q_x = -y^2 \quad P_y = 2xy - 3y^2$$

$$\frac{Q_x - P_y}{P} = \frac{y^2 - 2xy}{xy^2 - y^3} = \frac{2(y-x)}{y(x-y)} = -\frac{2}{y}$$

$$\mu = e^{-\int \frac{2}{y} dy} = e^{\ln y^{-2}} = \frac{1}{y^2}$$

$$\mu P dx + \mu Q dy = 0 \quad \text{exact}$$

$$u = \int \mu P dx + C(y) =$$

$$= \int (x-y) dx + C(y) = \frac{1}{2} x^2 - xy + C(y).$$

$$u_y = -x + C'(y) = \frac{1 - xy^2}{y^2} = \frac{1}{y^2} - x$$

$$C'(y) = \frac{1}{y^2} \Rightarrow C = -\frac{1}{y}$$

$$u = \frac{x^2}{2} - xy - \frac{1}{y}$$

$\mu$

$$u) \quad y(x^2 + y^2 + 1) - x(x^2 + y^2 - 1) dx = 0$$

$$\mu = \mu(xy)$$

$$P_y = x^2 + 3y^2 + 1$$

$$\mu P dx + \mu Q dy = 0$$

$$Q_x = -3x^2 - y^2 + 1 \quad \mu = \mu(xy)$$

$$(\mu P)_y = x\mu'P + \mu P_y$$

$$(\mu Q)_x = y\mu'Q + \mu Q_x \quad \left. \begin{array}{l} (\mu P)_y = x\mu'P + \mu P_y \\ (\mu Q)_x = y\mu'Q + \mu Q_x \end{array} \right\} + \mu'(xP - yQ) + \mu(P_y - Q_x) = 0$$

$$\frac{\mu'}{\mu} = \frac{-P_y + Q_x}{xP - yQ} = \frac{-4x^2 - 4y^2}{xy(2x^2 + 2y^2)} = \frac{-2}{xy}$$

$$z = xy \quad \frac{\mu'(z)}{\mu(z)} = \frac{-2}{z} \int$$

$$\frac{d\mu}{\mu(z)} = -\frac{2dz}{z} \int$$

$$|\mu(z)| = |z|^{-2}$$

$$\mu(z) = \frac{1}{z^2} = \frac{1}{(xy)^2}$$

$$u = \int \mu P dx + \mu Q dy = \int \frac{y(x^2 + y^2 + 1)}{x^2 y^2} dx + C(y)$$

$$= \int \frac{1}{y} + \frac{x}{x^2 y} + \frac{1}{x^2 y} = \frac{x}{y} - \frac{y}{x} - \frac{1}{xy} + C(y)$$

$$u_y = -\frac{x}{y^2} - \frac{1}{x} + \frac{1}{y^2 x} + C'(y) = \dots$$

$$4) \quad y dx - x dy = 2x^3 \tan\left(\frac{y}{x}\right) dx$$

$$z = \frac{y}{x} \quad dz = \frac{dy}{x} - \frac{y}{x^2} dx$$

$$dF = F_x dx + F_y dy$$

$$-x^2 dz = -x dy + y dx$$

$$-dz = 2x \tan(z) dx$$

$$-\frac{dz}{\tan(z)} = 2x dx$$

$$-\int \frac{\cos(z)}{\sin(z)} dz = \int 2x dx$$

$$-\ln|\sin(z)| = x^2 + C$$

$$-\ln \sin \frac{y}{x} = x^2 + C$$

$$C = -x^2 - \ln \left| \sin \frac{y}{x} \right| = u(x, y) = \text{priv. integr.}$$

Parametrično reševanje impl:itno podanih DE

Do zdaj so enačbe bile dolžne  $y' = F(x, y)$

lahko pa je v  $F(x, y, y') = 0$

Poanta:  $F(x, y, p) = 0$  podaja ploščo v  $\mathbb{R}^3$   
(če  $F \in C^1$  in  $\nabla F \neq 0$ )

Vseke rešitev bo potekala po neki krivulji:

$$\gamma(t) = (x(t), y(t), p(t)) \text{ da je } F(\gamma(t)) = 0$$

Da  $\gamma$  res podaja rešitev DE v parametrični obliki,  
mora veljati

$$\dot{y}(t) = p(t) \cdot \dot{x}(t)$$

Preprost primer:

$$F(x, y) = 0$$

↯

$F(x, p)$  padeja krivuljo v ravnini:

To krivuljo parametriziramo in poiščemo manjkajoči  
predpis za  $y$  komponento

6) Reči de

$$a) (y')^2 - 2y^2 = 1$$

$$b) x^2 + 1 + (y')^2 - 2y' = 1$$

$$p^2 - 2y^2 = 1$$

$$p = \operatorname{ch} t$$

$$y = \frac{1}{\sqrt{2}} \operatorname{sh}(t)$$

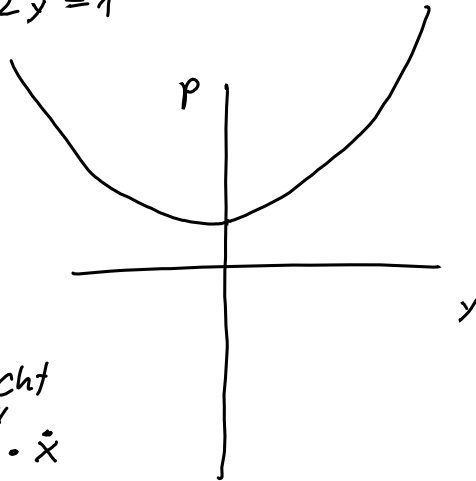
$$\dot{y} = \frac{\operatorname{ch}(t)}{\sqrt{2}} = \overset{\operatorname{ch} t}{p} \cdot \dot{x}$$

$$\dot{x} = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{2}{\sqrt{2}} t + C \quad t = \sqrt{2} x$$

kurve je  $(\frac{\sqrt{2}}{2} t + C, \frac{\sqrt{2}}{2} \operatorname{sh} t)$

$$y = \frac{\operatorname{sh}(\sqrt{2}(x+C))}{\sqrt{2}}$$

3C druga  $\Rightarrow$  5ta  $\frac{\operatorname{sh}(\sqrt{2}(C-x))}{\sqrt{2}}$  druga wirbe



$$b) x^2 + 1 + (y')^2 - 2y = 1$$

$$x^2 + (p-1)^2 = 1$$

$$x = \cos t$$

$$p = 1 + \sin t$$

$$\dot{y} = p \dot{x} = (1 + \sin t)(-\sin t) = -\sin^2 t - \sin t$$

$$y = \int \dot{y} dt = \int \frac{1 - \cos 2t}{2} dt + \cos t =$$

$$= \frac{1}{2} \cancel{t} \frac{1}{2} \sin 2t + \underbrace{\cos t}_x$$

$$y = \frac{1}{2} \arccos x - x \sin(\arccos x) + x$$

9.2.17: p'neri

$$a): p^2 - 2y^2 = 1$$

$$\text{er } F_p = 0 \quad F_y = 2p = 0 \Leftrightarrow p = 0$$

$\Rightarrow y$  konstante

$$y^2 = -\frac{1}{2} \quad \text{ni: keine reelle Lösung}$$

b)

Od združic: Parametrizirane reševanje DE  
oblike  $F(x, y, y') = 0$

11.11

Splosni postopek

1.  $p = y'$

2. izračunamo  $DF(x, y, \underbrace{p}_{\text{neodvisne spremenljivke}})$

3. s pomočjo nastanka  $dy = p dx$  dobimo

diferencialno enačbo  $x = x(p)$  ali  $y = y(p)$ .

Manjšo komponento dobimo iz prvoe enačbe

Včasih ima DE tudi singularne rešitve

1)  $G(x, y, C) = 0$  in enakosti  $\frac{\partial G}{\partial C} = 0$  eliminiramo  $C$

2) parametriziramo množico  $\{F = F_p = 0\}$  in preverimo  
ali reši DE  $\uparrow$  odvod po  $p$

1. naloga: Poišči splošno in merbitne sp. rešitve

$$x^2(y')^3 - xy' + y = 0$$

$$y = xy' - x^2(y')^3 \quad p = y' \quad dy = p dx$$

$$y = xp - x^2 p^3$$

$$dy = p dx + x dp - 2xp^3 dx - 3x^2 p^2 dp$$

$$\cancel{p dx} = \cancel{p dx} - 2xp^3 dx + (x - 3x^2 p^2) dp$$

$$2xp^3 dx = (x - 3x^2 p^2) dp$$

$$x=0 \Rightarrow y=0$$

$$x \neq 0 \Rightarrow$$

$$2p^3 dx = (1 - 3xp^2) dp$$

$$2p^3 x'(p) = 1 - 3xp^2$$

$$p=0 \Rightarrow y=0$$

$$\dot{x} = \frac{1}{2p^3} - \frac{3}{2} \frac{x}{p^2}$$

$$\dot{x} + \frac{3}{2} \frac{x}{p^2} = \frac{1}{2p^3}$$

homogeni del:

$$\frac{dx}{x} = -\frac{3}{2} \frac{dp}{p}$$

$$|x| = C |p|^{-\frac{3}{2}}$$

partikularni:

$$C = C(p)$$

$$\dot{x} = C(p) p^{-\frac{3}{2}} = \frac{1}{2} p^{-3}$$

$$\int_0^r \frac{1}{2} p^{\frac{3}{2}-3} = \frac{1}{2} \int_0^r p^{-\frac{3}{2}} dx = -p^{-\frac{1}{2}}$$

$$C(p) = -p^{-\frac{1}{2}}$$

$$x = C p^{-\frac{3}{2}} - p^{-2}$$

$$y = xy' - x^2(y')^3 = C p^{-\frac{1}{2}} - p^{-1} - (C^2 p^{-3} - 2C p^{-\frac{3}{2}-2} + p^{-4}) p^3$$

$$= C p^{-\frac{1}{2}} - p^{-1} - C^2 + 2C p^{-\frac{1}{2}} - p^{-1} =$$

$$= 3C p^{-\frac{1}{2}} - 2p^{-1} - C^2$$

Dobili smo rešitev v parametrični obliki  $(xp), y(p)$

$$\text{kjer je } p = \frac{y'(p)}{x'(p)} = y'(x)$$

Se singularne rešitve

$$x = -p^{-2} + D p^{-\frac{3}{2}}$$

$$y = -2p^{-1} + 3D p^{-\frac{1}{2}} - D^2$$

$$F(x, y, p) = y - xp + x^2 p^3 = 0$$

$$\text{torej } F_p = y - x + 3x^2 p^2 = 0 \Rightarrow x(1 + 3xp^2)$$

$x=0 \Rightarrow y=0$  ena točka ni a priori rešitev

$$1 + 3xp^2 = 0 \Rightarrow p^2 = -\frac{1}{3x} \Rightarrow p = \pm \sqrt{-\frac{1}{3x}}$$

$x < 0$

$$y - x\left(\pm \sqrt{-\frac{1}{3x}}\right) + x^2 \frac{1}{3x} \left(\pm \sqrt{-\frac{1}{3x}}\right) = 0$$

$$\text{torej } y = \pm \sqrt{-\frac{1}{3x}} \left(x - \frac{1}{3}x\right) = \pm \sqrt{-\frac{1}{3x}} \cdot \frac{2}{3}x = \pm \frac{2}{3} \sqrt{-\frac{x}{3}}$$

$$y' = \pm \frac{2}{3} \cdot \frac{\frac{1}{2} \sqrt{-\frac{1}{3x}}}{\sqrt{-\frac{x}{3}}} = \pm \frac{1}{3} \sqrt{-\frac{3}{x}}$$

Ali resimo enačbo

$$\pm \frac{2}{3} \sqrt{-\frac{x}{3}} - x \left(\pm \frac{1}{3} \sqrt{-\frac{3}{x}}\right) + x^2 \left(\pm \frac{1}{9} \frac{3}{x} \cdot \sqrt{-\frac{3}{x}}\right) = 0$$

$$= \pm \frac{2}{3} \sqrt{-\frac{x}{3}} \mp \frac{1}{3} \sqrt{3x} \pm \frac{3}{93} \sqrt{3x} \neq 0$$

$$= \pm \frac{2}{3} \sqrt{-\frac{x}{3}} \mp \frac{78}{93} \sqrt{3x} = \sqrt{x} \left(\pm \frac{2}{3\sqrt{3}} \mp \frac{78\sqrt{3}}{93}\right) =$$

$$= \sqrt{3x} \left(\pm \frac{2}{9} \dots\right)$$

2. način primerjamo  $y'$  in  $p$

$$\pm \frac{1}{3} \sqrt{-\frac{3}{x}} = \pm \sqrt{-\frac{1}{3x}}$$

$$\pm \frac{\sqrt{3}}{3\sqrt{x}} = \pm \frac{1}{\sqrt{3x}} \quad \times$$

2. nalez

Dobro je rešitve

$$(y')^2 - xy' - y + \frac{x^2}{2} = 0$$

$$y = (y')^2 - xy' + \frac{x^2}{2}$$

$$y = p^2 - xp + \frac{x^2}{2}$$

$$dy = 2p dp - x dp - p dx + x dx \quad dy = p dx$$

$$2p dx - x dx = 2p dp - x dp$$

$$(2p - x) dx = (2p - x) dp$$

$$1. \quad 2p - x = 0$$

$$2p = x \Rightarrow p = \frac{x}{2}$$

$$y = p^2 - 2p^2 + 2p^2 = p^2$$

$$2. \quad 2p - x \neq 0$$

$$dx = dp$$

$$p'(x) = 1$$

$$p(x) = x + C$$

$$\begin{aligned} y &= 1 - x + \frac{x^2 + 2xC + C^2}{2} = \\ &= \frac{x^2 + 2x(C-1) + C^2 + 2}{2} = \\ &= \frac{x^2}{2} + x(C-1) + C^2 + 2 \end{aligned}$$

$$y = \frac{x^2}{2} + xC + C^2$$

Me

$$0 = x + 2C \Rightarrow C = -\frac{x}{2}$$
$$\frac{x^2}{2} - \frac{x^2}{2} - x + \frac{x^2}{4} + 2$$

singularne:

$$0 = x + 2C$$

$$C = -\frac{x}{2}$$

$$\Rightarrow y = \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{4} = \frac{x^2}{4}$$

## Clairautova enačba

$$y = xy' + \psi(y')$$

Videl: bomo, da ogrinječo najdemo že sproti

3. naloga

$$a) y = xy' + \frac{1}{y'}$$

$$b) y = xy' + (y')^2 + 1$$

$$a) y = xy' + \frac{1}{y'}$$

$$\psi(t) = \frac{1}{t}$$

$$y' = p$$

$$y = px + \frac{1}{p}$$

$$dy = p dx + x dp - \frac{1}{p^2} dp \quad p dx = dy$$

$$\frac{1}{p^2} dp = x dp$$

$$0 = \left(\frac{1}{p^2} - x\right) dp$$

Clairautova enačba se vedno reducira na obliko  $f(x, p) dp = 0$

$$1) dp = 0 \Rightarrow p = c$$

$$y = xc + \frac{1}{c}$$

$dp = 0 \dots$  splošna rešitev

$f(x, p) = 0 \dots$  ogrinječa

$$2) \frac{1}{p^2} = x \Rightarrow p = \pm \sqrt{\frac{1}{x}}$$

$$y = \pm \sqrt{x} + \pm \sqrt{x} = \pm 2\sqrt{x}$$

$$y' = \mp \frac{1}{\sqrt{x}}$$

$$b) y = xp + p^2 + 1$$

$$dy = xdp + p dx + 2p dp \quad dy = p dx$$

$$(x + 2p) dp = 0$$

$$dp = 0 \Rightarrow p \equiv C \Rightarrow y = xC + C^2 + 1$$

$$x = -2p \Rightarrow p = -\frac{1}{2}x \Rightarrow y = -\frac{1}{2}x^2 + \frac{1}{4}x^2 + 1 = -\frac{1}{4}x^2 + 1$$

$$y' = -\frac{1}{2}x$$

Opomba o singularnih rešitvah Clairotovih enačb:

$$y = xy' + \psi(y')$$

$$F(x, y, p) = xp + \psi(p) - y = 0$$

$$F_p = x + \psi'(p) = 0$$

Splazna rešitev

$$0 = G(x, y, c) = xC + \psi(c) - y = F(x, y, c) = 0$$

Torej odvajanje v  $p$  ti vedno da ~~ta~~ ogrinjajo  
pri Clairotovi enačbi

Lagrangeva enačba

$$y = x \varphi(y') + \psi(y')$$

4. Reži

$$y = \frac{xy'}{2} + \frac{2}{y'} \quad \varphi(y') = \frac{y'}{2}$$

$$dy = \frac{x}{2} dp + \frac{p}{2} dx - \frac{1}{p^2} dp \quad dy = \cancel{dp} p dx$$

$$\frac{1}{2} p dx = \frac{x}{2} dp - \frac{1}{p^2} dp$$

$$p dx = \left(x - \frac{2}{p^2}\right) dp$$

$$\dot{x} = \frac{dx}{dp} = \frac{x}{p} - \frac{2}{p^3} \quad \text{LDE za } x = x(p)$$

$$\dot{x} - \frac{x}{p} = -\frac{4}{p^3}$$

homogeno:

$$\frac{dx}{x} = \frac{1}{p} dp$$

parti:

$$|x| = C |p| \quad x = C \cdot p$$

$$C'(p) \cdot p = -\frac{4}{p^3} \Rightarrow C'(p) = -\frac{4}{p^4}$$

$$C = -\frac{4}{3} p^{-3}$$

$$\frac{x}{p} = 12 \cdot p^{-3} \cdot p + C \cdot p \Rightarrow \frac{x}{p} = \frac{12}{p^2} + C \cdot p$$

$$y = \frac{1}{2} \left( \frac{12}{p^2} + C p^2 \right) + \frac{2}{p} = \frac{1}{p} \left( \frac{2}{3} + \frac{1}{2} \right) + \frac{C}{2} p^2 = \frac{8}{3} + \frac{C}{2} p$$

Singularna rešitev:  $0 = \frac{x}{2} - \frac{2}{p^2} = F_p$

$$\frac{x}{p^2} = \frac{x}{2}$$

$$p^2 = \frac{4}{x}$$

$$\Rightarrow y = \frac{2}{p^2} \cdot p + \frac{2}{p} = \frac{4}{p}$$

Ali: je res  $dy = p dx$ ?

$$\dot{y} = -\frac{4}{p^2}$$

Ali: velja  $\dot{y} = p \cdot \dot{x}$

$$\dot{x} = -\frac{8}{p^3}$$

$$\cancel{\frac{4}{p^2}} \quad p \cdot \left(-\frac{8}{p^3}\right) = -\frac{8}{p^2} \neq -\frac{4}{p^2}$$

$\Rightarrow$  To ni singularna rešitev

5. Resi  $y = \frac{x(y')^2}{2} + \frac{2}{y'}$ .



;

## Povzetek (o reševanju impl. DE)

$F(x, y, y') = 0$   $\nabla F \neq 0$   $F$  dovolj gladka  
 $F(x, y, p)$  je ploskev

$$P(x_0, y_0, p_0) \in \mathbb{R}^3$$

$$1) F_p(P) \neq 0 \Rightarrow p = p(x, y)$$

$$F(x, y, p(x, y)) = 0 \text{ za } x, y \text{ blizu } x_0, y_0$$

$$p(x_0, y_0) = p_0$$

če je  $x \mapsto y(x)$  rešitev začetnega problema  $F(x, y, y')$   
 $y(x_0) = y_0$

$$\text{Potem je } y' = p(x, y)$$

tako dobimo splošno rešitev

$$2) F_p(P) = 0 \Rightarrow$$

dobimo množico  $\{F = F_p = 0\} \leftarrow$  unija točk in krivulj

če je  $P_0$  izolirana točka:  
na neki okolici  $U$  od  $P_0$  velja

$$\nabla F \neq 0 \text{ za } P \in U - \{P_0\}$$

dobimo splošno rešitev, vendar se  
lahko zgodi, da več čenic splošne rešitve poteka  
skozi  $P_0$

če je  $\{F = F_p = 0\}$  krivulja, jo parametiziramo

npr. kot  $(x(p), y(p), p)$ ; če velja  $p \cdot \dot{x}(p) = \dot{y}(p)$ ,

potem je to singularna rešitev (in hkrati

ogrinjajoča splošne rešitve

# Primer od zadnjic

18.11

$$y = \frac{xy^2}{2} + \frac{2}{y}$$

1)  $F_p = 0$

$$F(x, y, p) = \frac{xp^2}{2} + \frac{2}{p} - y$$

$$F_p = 0 \Rightarrow xp - \frac{2}{p^2} = 0 \Rightarrow x = \frac{2}{p^3}$$

$$y = \frac{xp^2}{2} + \frac{2}{p} = \frac{2}{p^3} \cdot \frac{p^2}{2} + \frac{2}{p} = \frac{3}{p}$$

$$\dot{y} = -\frac{3}{p^2}$$

$$p\dot{x} = p \left(-\frac{6}{p^3}\right) = -\frac{6}{p^2} \neq \dot{y} \Rightarrow \text{G ni agirirnjace}$$

oz ni sing. res

2)  $dy = p dx$

$$dy = xp dp + \frac{p^2}{2} dx - \frac{2}{p^2} dp$$

$$\left(\frac{p^2}{2} - p\right) dx + \left(xp - \frac{2}{p^2}\right) dp = 0$$

Ker nas zanimajo rešitve v obliki grafov funkcij  $x \mapsto y(x)$  lahko predpostavimo, da  $dx \neq 0$

$\Rightarrow$  formalno lahko delimo z  $dx$ ; dobimo DE za

$$p = p(x).$$

Običajno je lažje rešavati DE za  $x = x(p)$

želimo deliti z  $dp$ , da dobimo  $\frac{dx}{dp} = \dot{x}(p)$

Ali je lahko  $dp = 0$ ?  $\Rightarrow p \equiv C = y \Rightarrow y(x) = Cx + D$  za ustrezen  $D$

$$\text{Sicer} \Rightarrow \left(\frac{p^2}{2} - p\right) \frac{dx}{dp} + xp - \frac{2}{p^2} = 0$$

$$\text{Opazimo: } dp = 0 \Rightarrow \frac{p^2}{2} - p = 0$$

$$dp \neq 0 \Rightarrow \text{dobimo DE, če } \frac{p^2}{2} - p \neq 0$$

$$\text{če } \frac{p^2}{2} - p = 0 \Rightarrow p = 0 \vee p = 2$$

ni definirana  $\rightarrow y = 2x + 1$

Ali je resitev singularna?

$$F_p = xp - \frac{2}{p^2}$$

Premica  $y = 2x + 1$  ustreza krivulji:

$$x \mapsto (x, y(x), y'(x)) =$$

$$(x, 2x + 1, 2)$$

$$F_p = 2x - \frac{1}{2} \Rightarrow$$

Povsod razen v  $x = \frac{1}{4}$  je ta izraz  $\neq 0$

Ta premica torej pripada splošni rešitvi:

$x = \frac{1}{4}$  pa je singularna točka

lahko

smo pa iskali:  
zadnjic

## Povzetek pavzeta

- 1)  $F_p = 0$ , parametriziramo/eliminiramo  $p$  iz enačb  
 $F = F_p = 0$ , pravimo če je rešitev  $\rightarrow$  singularna
- 2) Splošno rešitev iščemo s pomočjo nastanka  $dy = p dx$

Naj bo  $y = \Phi(x, c)$  splošna rešitev DE  $F(x, y, y') = c$   
in  $\gamma$  ogrinjača. Pokaži da skozi vsako točko na grafu  $\gamma$   
poteka neskončno mnogo rešitev

DN

Reži DE

$$y = xp^2 - 2p^3$$

$$y = (xy'^2) - 2(y')^3$$

$$dy = p dx = p^2 dx + 2p x dx - 6p^2 dp$$

$$dx(p^2 + p) + 6p^2 dp = 0$$

$$1) dp = 0 \Rightarrow p = c, y = c^2 x - 2c^3$$

$$y' = c^2 = p = c$$

$$c^2 = c \Rightarrow c = 0 \vee c = 1$$

$$\downarrow \\ y = c \\ \text{sing.}$$

$$\hookrightarrow y = x - 2 \\ \text{ni singularna}$$

$$F_p = F = 0$$

$$-2xp + 6p^2 = 0$$

$$-2p(x - 3p) = 0$$

$$p = 0 \Rightarrow$$

$$x = 3p \Rightarrow$$

$$F = 0 \Rightarrow 3p^3 - 2p^3 = 0 \\ p^3 = 0$$

$$\dot{x}(p) = 3$$

$$\dot{y}(p) = 3p^2$$

$$p \dot{x} \neq \dot{y} \Rightarrow \text{to ni rešitev}$$

$$dp \neq 0 \Rightarrow$$

$$\dot{x}(p - p^2) + 6p^2 - 2px = 0 \quad / : p(1-p) \quad \leftarrow \text{tako že obravnava,} \\ \text{ko je to 0}$$

$$\dot{x} + x \frac{-2p}{p-p^2} = \frac{6p^2}{p^2-p}$$

homogeni del:

$$\frac{dx}{x} = \frac{-2p dp}{p-p^2} = \frac{-2}{1-p} dp$$

$$\ln|x| = -2 \ln|1-p|$$

$$x = D(1-p)^2$$

partikularni:

$$D'(1-p)^{-2} = \frac{6p}{p-1}$$

$$D' = 6p(p-1) = 6p^2 - 6p$$

$$D = 2p^3 - 3p^2 + C$$

$$\text{Splošno: } x = \frac{(2p^3 - 3p^2 + C)}{(1-p)^2}$$

Enadbe višjega reda

$$F(x, y^{(k)}, \dots, y^{(n)}) = 0 \quad 0 \leq k \leq n$$

Lahko znižamo red z nastavitvami:  $z = y^{(k)}$

$$\hookrightarrow F(x, z, \dots, z^{(n-k)}) = 0$$

$$xy''' = y''$$

$$z = y''$$

$$xz' = z$$

$$\frac{dz}{z} = C \frac{dx}{x} \quad \leadsto |x| = |z| \Rightarrow x = |z| = |y''|$$

$$z = C \cdot x = y''$$

↑ ± skrajšamo v konstanto

$$\begin{aligned} \int \int (Cx dx) dx &= \int C \frac{1}{2} x^2 dx = \int Cx^2 dx = C \frac{1}{3} x^3 + Dx + E = \\ &= Cx^3 + Dx + E \end{aligned}$$

②

$$F(y, y', \dots, y^{(k)}) = 0 \hookrightarrow F \text{ neodvisen od } x$$

$$''z(y) = y'(y)'' \quad \text{zod } y \text{ je y odvod od } y$$

Interpretiramo kot:

$$z(y) = y'(x(y)) \quad z(y(x)) = y'(x)$$

$y \mapsto x(y)$  inverz razitve  $y$   
(obšteja ker  $y'(x) \neq 0$ )

Reš enačbo

$$y y'' = y'(y' + 1)$$

$$y' = 0 \Rightarrow y y'' = 0$$

Možnost ko ne moremo uporabiti  
nabivke

$$1) y = 0$$

$$2) y'' = 0 \Rightarrow$$

$$y' = C = 0 \Rightarrow y = C$$

$$\Rightarrow y = C \in \mathbb{R}$$

$$z = y'(x(y))$$

$$z(y(x)) = y'(x)$$

$$\frac{dz}{dy} \cdot y' = y''(x) \quad \dot{z} y' = y''$$

$$\dot{z} \cdot z = y''$$

Naslednja enačba

$$y \cdot \dot{z} \cdot z = z(z + 1)$$

$$1) z = 0$$

$$2) z \neq 0 \Rightarrow \text{lahko delimo}$$

$$2) \quad y \dot{z} = z + 1$$

$$\text{ham: } \frac{dz}{dy} \cdot y = z \quad z = Cy$$

partikularne:

$$y \cdot C' y = 1$$

$$C' = \frac{1}{y^2} \leadsto C = -\frac{1}{y} + D$$

$$z = -1 + D y$$

$$\frac{y'}{-1 + D y} = 1$$

$$x = \int \frac{dy}{-1 + D y} = \frac{1}{D} \ln(-1 + D y) + C$$

$$D x + C = \ln(D y - 1)$$

$$\frac{C e^{D x} + 1}{D} = y$$

$$D = 0 \Rightarrow$$

$$\frac{y'}{-1} = 1$$

$$y' = -1$$

$$y = z - x$$

$$F(x, y, y', \dots) = 0$$

$F$  homogena v  $y, y', \dots$

$$F(x, \lambda y, \lambda y', \dots) = \lambda^k F(x, y, y', \dots) \quad \text{za nek } k \in \mathbb{R}$$

$k$  in za  $\forall \lambda \in \mathbb{R}$

$$\dots \text{ uvedemo } z(x) = \frac{y'}{y}$$

$$yy'' - x^2 y^2 + (y')^2$$

$$z = \frac{y'}{y} \quad yz = y' \quad y'' = y'z + z'y = z^2 y + z'y$$

$$y(yz^2 + z'y) = x^2 y^2 + (yz)^2$$

$$y = 0 \Rightarrow \text{je rešitev}$$

$$y \neq 0 \Rightarrow$$

$$\cancel{y}z^2 + z'y = x^2 + \cancel{yz}z$$

$$z = \frac{1}{3}x^3 + C = \frac{y'}{y} \Rightarrow \ln|y| = \frac{x^4}{12} + Cx + D$$

$$y = D e^{\frac{x^4}{12} + Cx}$$

## Ekstremni izreki

$f: D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  zvezna in Lipschitzova v  $y$

$$|f(x, y_0) - f(x, y_1)| < L|y_0 - y_1| \quad \text{za } L \geq 0$$

$\Rightarrow \exists!$  rešitev  $y' = f(x, y)$ ,  $y(x_0) = y_0 \quad \forall (x_0, y_0) \in D$ , ki je definirana na intervalu  $(x_0 - \alpha, x_0 + \alpha)$ , kjer je

$$\alpha = \min \left\{ a, \frac{b}{M}, \frac{1}{L} \right\}, \text{ kjer sta } a, b > 0 \text{ takšna, da je}$$

$$\underbrace{[x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]}_{C_{a,b}} \subset D \quad \text{in je } M = \max_{x,y \in C_{a,b}} |f(x,y)|$$

obampli:  $f \in C^1[a,b] \Rightarrow f$  je Lipschitzova

Podana je Cauchyjeva naloga

$$y' = y + \cos y \quad y(0) = 0$$

a) Utemelji da obstaja natanko ena rešitev in določi

Taylorjev polinom stopnje 3 v  $x=0$

b) določi  $\alpha$

c) Dokazi, da se rešitev razširi na  $\mathbb{R}$

d) Dokazi da velja  $\lim_{x \rightarrow \infty} y(x) = \infty$

a) F je zveza na  $\mathbb{R}^2$

$f$  je Lipschitzeva? da ker je  $v C^1$

$\Rightarrow$  Rešitev obstaja in je enolično določena

$$y'' = y' - \sin(y)y' = (y + \cos y)(1 - \sin(y))$$

$$\begin{aligned} y''' &= (y' - y' \sin(y))(1 - \sin(y)) + (y + \cos y)(1 - \cos(y)y') = \\ &= (y + \cos y)(1 - \sin(y))^2 + (y + \cos(y))(1 - y \cos y + \cos^2 y) \end{aligned}$$

$$\begin{aligned} y(x) &= y(0) + y'(0)x + \frac{1}{2} y''(0)x^2 + \frac{1}{6} y'''(0)x^3 + o(x^3) = \\ &= x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + o(x^3) \end{aligned}$$

$$b) \quad L = \max_{C(a,b)} \left| \frac{\partial f}{\partial y} \right|$$

$$f(x,y) = y + \cos y$$

$$\left| \frac{df}{dy} \right| = |1 - \sin(y)| \leq 2 \quad L=2$$

$$\alpha = \min(a, \frac{b}{M}, \frac{1}{2})$$

$$\max_{C(a,b)} f \stackrel{(*)}{=} \max_{[-b,b]} f(\cdot, y) = M$$

$$|f(x,y)| = |y + \cos y| \leq |y| + |\cos y| \leq b+1$$

$$\frac{b}{M} \geq \frac{b}{b+1}$$

$$\alpha \geq \min \left\{ a, \frac{b}{b+1}, \frac{1}{2} \right\}$$

če izberemo  $a$  in  $b$  dovolj velika, je

$$\text{lahko } \alpha > \frac{1}{2}$$

$$\Rightarrow \text{pa} \text{ } \alpha \text{ ne } \in (-\frac{1}{2}, \frac{1}{2})$$

c) Pogoj iz b) velja za vsako točko.  
Torej če si postopoma izbiramo točke vedno  
bolj na robovih že manjše, lahko  
razširimo intervale na cel  $\mathbb{R}$   
razen namreč b) sproti postane b) + b)

Sklep: Če lahko  $\alpha$  omejitve nevedel neodvisno  
od začetnega pogoja, potem se rešitev  
razširi na  $\mathbb{R}$

V posebnem to velja: če sta  $f$  in  $f_y$  omejena

$$d) \lim_{x \rightarrow 0} y(x) = \infty$$

$y$  narašča po vsej od 1

$$y' > 0 \text{ velja za } t > 1$$

$$1 - \sin t \geq 0 \text{ za } t > 0 \Rightarrow \text{fun narašča}$$

Dokazano da je  $y(x) > 0 \forall x > 0$

Reimode  $y(\tilde{x}) = 0$  za nek  $x' > 0$

$$y'(\tilde{x}) = 0 + \cos(0) = 1$$

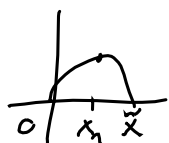
Torej je mogla biti prej funkcija negativna,

$$Z = \{x, > 0, x(x) = 0\}$$

$$y_0 = 1 \quad y = f(x)$$

$$\Rightarrow f(x) > 0$$

$$\text{za } x \in (0, \epsilon) \quad \epsilon > 0$$



$$\forall \epsilon > 0: x_1 \in (0, \tilde{x})$$

$$f'(x_1) = 0$$

$$f'(x_1) = f(x_1) + \cos(f(x_1)) = 0$$

$$\Rightarrow \cos f(x_1) < 0$$

$$f(x_1) > \frac{\pi}{2}$$

$$f'(x_1) > \frac{\pi}{2} + \cos(\lambda_1) > 0 \quad *$$

$$\Rightarrow f(x) > 0 \text{ za } \forall x > 0$$

$$\Rightarrow f'(0) \text{ jo naraščajoča}$$

## Eksistenčni izreki

$f: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  zvezan in Lipschitzova

$\Rightarrow \exists!$  rešitev  $y = f(x, y)$   $y(x_0) = y_0$  za  $\forall (x_0, y_0)$ ,  
ki je definirana na intervalu  $(x_0 - \alpha, x_0 + \alpha)$ ,  
kjer je  $\alpha = \min \left\{ a, \frac{b}{M}, \frac{1}{L} \right\}$ ;  $M = \max |f|$ ,  $L$  Lipschitz,  
 $a, b > 0$ .  $[x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] \subset D$

$f \in C^1[a, b] \Rightarrow f$  je Lipschitzova

---

$$F(x, y, y') = 0$$

F homogena v  $y, y'$  ( $F(x, \lambda y, \lambda y') = \lambda^k F(x, y, y')$ )

$\Rightarrow$  uvedemo  $z(x) = \frac{y'}{y}$

---

Enačba višjega reda  $F(x, y^{(k)}, \dots, y^{(n)}) \Rightarrow z = y^{(k)}$

---

Reševanje impl. DE

$$F(x, y, y') = 0 \quad \nabla F \neq 0$$

1)  $F(x, y, p) = 0$  je ploskev

$$P = (x_0, y_0, z_0)$$

2.1)  $F_p(P) \neq 0 \Rightarrow p = p(x, y)$

$$F(x, y, p(x, y)) = 0 \text{ v okolici } P$$

$$p(x_0, y_0) = p_0$$

$$y(x) \text{ rešitev} \Rightarrow y' = p(x, y)$$

Dobimo **splošno rešitev**

2.2)  $F_p(P) = 0 \Rightarrow \{F = F_p = 0\}$  je unija točk in krivulj

$P_0$  izolirana točka  $\Rightarrow \exists$  okolica  $P$ .  $P \in U - \{P_0\}$ .  $\nabla F \neq 0$

Dobimo **splošno rešitev**

$\{F = F_p = 0\}$  je krivulja  $\Rightarrow$  parametriziramo  
npr.  $(x(p), y(p), p)$ .

$$p \cdot \dot{x}(p) = \dot{y}(p) \Rightarrow \text{Singularna rešitev}$$

  
Ogrinjača **splošne rešitve**

## Clairautova enačba

$$y = xy' + \psi(y') \leadsto y' = p \leadsto p dx = dy$$

$$\text{Dobimo: } f(x, p) dp = 0$$

$$dp = 0 \dots \text{singularna rešitev}$$

$$f(x, p) = 0 \dots \text{ogrinjača}$$

Odvajanje po  $p$ , ti vedno da ogrinjačo pri tej enačbi:

---

Prvi integral enačbe  $y' = f(x, y)$  je funkcija, ki je konst.

vzdolž vsake rešitve

$$y = \varphi(x, C) \wedge \varphi_C \neq 0 \Rightarrow C = C(x, y)$$

$$1) \text{ DE zapišemo ko } P dx + Q dy = 0$$

$$Q_x = \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = P_y \Rightarrow (P, Q) \text{ je potencialno}$$

$$Q = \frac{\partial}{\partial y} \int P dx + C'(x, y)$$

---

## Riccatijeva enačba

$$y' = a^2 y + b y + c$$

$$1) \text{ Uganemo rešitev } y_p$$

$$2) \text{ Splošna rešitev je } y = y_p + u(x)$$

$$3) \text{ Dobimo Bernullijeva enačba } y = y_p + \frac{1}{u^2(x)} \Rightarrow \text{LDE}$$

---

## Bernullijeva enačba

$$y' + a y = b y^\alpha \quad \alpha \notin \{0, 1\} \leadsto z = y^{1-\alpha}$$

25.11

Od zadnjice naprej  $y' = y + \cos y$   $y(0) = 0$

Rešitev  $\varphi$  je definirana za  $\forall x \in \mathbb{R}$

$y > 0$  za  $\forall x \geq 0$

$$Z = \{x > 0, \varphi(x) = 0\}$$

$$\varphi'(0) = \varphi(0) + \cos(0) = 0 + \cos 0 = 1$$

$$\Rightarrow \text{za nek } \varepsilon > 0 \text{ je } \varphi(x) > 0 \quad \forall x \in (0, \varepsilon)$$

$$\text{označimo } x_0 = \inf Z > 0$$

$$\varphi \text{ zveča} \Rightarrow \varphi(x_0) = 0$$

$$\text{Rolle: } \exists x_1 \in (0, x_0), \varphi'(x_1) = 0 \wedge \varphi(x) > 0 \quad \forall x \in (0, x_0)$$

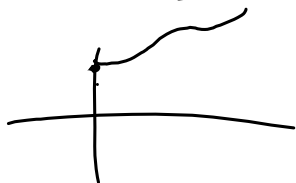
$$\varphi \text{ res} \text{ CN} \Rightarrow \varphi'(x_1) = \varphi(x_1) + \cos(\varphi(x_1))$$

$$\stackrel{V}{0} \Rightarrow \varphi(x_1) > \frac{\pi}{2}$$

$$\Rightarrow \varphi(x_1) - \cos(\varphi(x_1)) > 0$$

$$\Rightarrow Z = \emptyset \quad *$$

$$g(t) = t + \cos t \quad g(0) = 1 \quad g'(t) = 1 - \sin t \geq 0 \quad \forall t$$



$$g(t) \geq 1, \quad \forall t \geq 0$$

$$\varphi'(x) = g(\varphi(x)) \geq 1 \quad \forall x \geq 0$$

$$\varphi(x) = \varphi(0) + \int_0^x \varphi'(t) dt \geq \int_0^x 1 dt = x \xrightarrow{x \rightarrow \infty} \infty$$

1) Dane je Cauchyjeva naloga

$$y' = \frac{y}{1+x^2+y^2} \quad y(0)=1$$

Dokazi da obstaja ! rešitev, ki je definirana na  $\mathbb{R}$ .

Ali ima  $y$  ničlo?

---

$f(x,y)$  mora biti zvezna in Lipschitzova v  $y$

$$f(x,y) = \frac{y}{1+x^2+y^2} \quad \text{zveznost zaradi elementarnosti}$$

Naredimo pravokotnik  $[a, a] \times [-b, b]$

Ta  $m$  je Lipschitzova, ker je odvod omejen

$\Rightarrow$  obstaja snje endično deloena na  $[-\alpha, \alpha]$

$$\text{kjer je } \alpha = \min \left\{ a, \frac{b}{M}, \frac{1}{L} \right\}$$

$a, b$  poljubno velike

$$M = \max_{[a,b]} |f(x,y)| \stackrel{x^2=0}{=} \max \frac{y}{1+y^2} = \left\{ \pm \frac{1}{2} \right\} = \frac{1}{2}$$
$$\hookrightarrow \frac{(1+y^2) - 2y^2}{(1+y^2)^2} = \frac{1-y^2}{1+y^2}$$

$$L = \max \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y} = \frac{1+x^2+y^2-2y^2}{(1+x^2+y^2)^2} = \frac{1+x^2-y^2}{(1+x^2+y^2)^2}$$

$L$  je tudi omejena

Dobili smo  $\alpha$ , ki je isti za vse točke in neodvisen od začetnega pogoja, tako da tako z eksistenčnim rešenjem rešimo vse točke

5)  $y' = 0 \Leftrightarrow y = 0$

Recimo da je  $f(x_0) = 0$

$$z \equiv 0 \quad z' = f(x,y) \quad \text{reši} \quad z(x_0) = 0$$

$f$  tudi reši to nalogo  $\Rightarrow f = z$  (zaradi endičnosti)

$\rightarrow$  ,ker je  $f(0) = 1$

Trditev: Za  $\alpha$  lahko vzamemo  $\alpha = \min \{a, \frac{b}{H}\}$   
L mae pa obstajati na tem pravokotniku

2)

$f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  zvezu

$$y'(x) = 1 + \frac{y(x)^2}{2} \sin f(x) \quad ; x \in [0, 2]$$

$$y(0) = 0$$

Dokazi da je

$$\max_{x \in [0, 2]} y(x) \geq 1$$

Recimo da  $\max y(x) < 1 \Rightarrow$

$$y'(x) \geq 1 + \frac{y(x)^2}{2} \geq \frac{1}{2}, \forall y \in [0, 1]$$

$y \geq 0$

Recimo da  $\exists x_0 > 0$  da je  $y(x_0) = 0$

Potem po Rollovem izteku,  $\exists x_1 \in (0, x_0)$  da je  $y'(x_1) = 0$

$$y'(x_1) = 1 + \frac{(y(x_1))^2}{2} \sin(f(x_1))$$

$$0 \leq y(x_1) \leq 1$$

$$y'(x_1) \geq 1 - \frac{(y(x_1))^2}{2} \geq \frac{1}{2}$$

$\Rightarrow y > 0$  za  $\forall x \in \mathbb{R}_{\geq 0} \cap [0, 2]$ , iker je odred v 0 pozitiven

$$\Rightarrow y'(x) \geq 1 - \frac{1}{2} \geq \frac{1}{2}$$

$$\text{Potem je } y(2) = y(0) + \int_0^2 y'(t) dt \geq 0 + 2 \cdot \frac{1}{2} = 1 \quad *$$

$$\Rightarrow \max y \geq 1$$

3.  $y' = f(y) \quad f \in C^1(\mathbb{R})$

Pokaži da je  $\forall$  periodična rešitev konstantna

---

Recimo da  $\exists w > 0. y(x+w) = y(x) \forall x$

$$\int_0^w y' dx = \int_0^w f(y) dx = c$$

$y$  periodična  $\Rightarrow f(y)$  periodična

če je  $f(c) = 0$ , potem je  $y \equiv c$  rešitev

$f(y)$  je odvod  $\Rightarrow f$  ima ničlo (ker je  $y$  periodična)

$\Rightarrow \exists x_0 \in \mathbb{R}. f(y(x_0)) = 0$   $y \equiv y(x_0)$  je ena rešitev  
sistema in zaradi enoličnosti je to edina

4.

$$y' = y + e^y \quad y(0) = 1$$

a) Dokaži, da je definicijski interval rešitve, kot ga določi obstojni teoreti omejen

b) Dokaži, da je  $f$  naraščajoča v  $\forall$  točki  $x \geq 0$ , kjer je definirana

c) Dokaži, da ima  $f$  ničlo

d)  $\exists w. 0 < w < \infty. \lim_{x \rightarrow w} f(x) = \pm \infty$

a)  $y + e^y$  je vedno odvedljiva  $\Rightarrow \exists$  enolična rešitev

na  $[\alpha, -\alpha]$   $\alpha = \min\{a, \frac{b}{M}, \frac{1}{\lambda}\}$

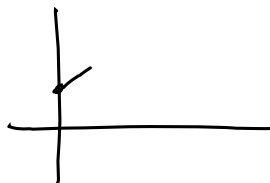
$$y \in [1-b, 1+b] \Rightarrow y + e^y \leq 1+b + e^{1+b}$$

$$M = 1+b + e^{1+b} \Rightarrow \frac{b}{M} = \frac{b}{1+b + e^{1+b}} \leq C \quad \forall b > 0$$

ta funkcija je omejena

b)  $y' = y + e^y$

$y' \geq 0 \quad y(0) = 1$



Rešimo da je  $y + e^y = 0$

$$e^y = -y$$

$\forall$   
 $0$

če je  $y=0$  pomeni, da  $y' = 1$  kar pomeni, da v vsaki točki narašča, kar pa ne more biti, ker rešen za eno ničlo

c) f me nide

Reano da nide nide. pden je  $f(x) > 0 \forall x \in \mathbb{R}$ ,

pden je p  $f$  nrašajoa

n odvod je veji od 1. pden pnda nide

skhati  $x$  os ali p nrašajoa nide ne marena  
razširiti če je p, ne more biti nrašajoa  
torej so p nrašajoa, pden je p arad

$< 0$  ✗

$$f(x) = f(c) + \int_c^x f'(t) dt \leq f(c) + x \xrightarrow{x \rightarrow \infty} -\infty$$

✓

d) Dokazujemo:

$$\exists \omega > 0. \lim_{x \rightarrow \infty} f(x) = \infty$$

Dokazmo da je inverz omejen. (inverz obstaja ker je  $f$   
strogo nrašajoa na  $x \geq 0$ )

$$\frac{1}{z'} = y + e^y \Rightarrow z' = \frac{1}{y + e^y}$$

$$z(y) = \int_0^y \frac{1}{t + e^t} dt \leq \int_0^y \frac{1}{e^t} dt = e^{-1} - e^{-y} \leq M$$

dostaja tukaj

↑  
 $t \geq 0$

Naj bo  $f \in C^1(\mathbb{R}^2)$ , zadošča predpostavkam eksistenčnega izreka. Recimo, da se naše rešitve  $y$  ne da razširiti cez interval  $(\alpha, \omega)$ ;  $-\infty < \alpha < x_0 < \omega < \infty$ , potem velja  $\lim_{x \rightarrow \omega} y(x) = \pm \infty$  in  $\lim_{x \rightarrow \alpha} y(x) = \pm \infty$

5)

$$y' = \frac{1}{x^2 + y^2} ; y(0) > 0$$

a) Pokaži, da  $\exists!$   $\varphi$  rešitev, ki je definirana na  $\mathbb{R}$

b) Pokaži  $\lim_{x \rightarrow \infty} \varphi(x) < \infty$

$f(x, y) = \frac{1}{x^2 + y^2}$  je omejena razen v točki 0

$f_y(x, y) = \frac{2y}{(x^2 + y^2)^2}$  je omejen razen na točki 0

Dokazimo da je rešitev omejene na vsakem končnem intervalu

v 0 je rešitev pozitivna in naraščajoča

$$\varphi(x) = \varphi(0) + \int_0^x \frac{1}{t^2 + (\varphi(t))^2} dt = \varphi(x_0) + \int_{x_0}^x \frac{1}{t^2 + (\varphi(t))^2} dt \leq$$

$$\leq \varphi(x_0) + \int_{x_0}^x \frac{1}{t^2} dt = \varphi(x_0) - \frac{1}{x} + \frac{1}{x_0} \leq M$$

$\uparrow$  omejeno

$\Rightarrow$  Po trditvi: je  $w = \infty$

Popravimo  $f$  tako, da nima več pda v (0,0) in velja  $f = \tilde{f}$  izven neke majhne okolice (0,0) ki je gost  $\varphi$  ne seka

v ~~tem~~  $\infty$  smer je simetrična  $\varphi(-x)$

6)

Pokažimo da je  $\varphi(x)$  naraščajoča in omejena  $\Rightarrow$  ima limito

Vektorsko polje na  $\mathbb{R}^n$  je preslikava  $V: \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

Tokovnica polja  $V$  je krivulja  $\gamma: (a,b) \rightarrow \mathbb{R}^n$ , ki  
 $\downarrow$   
zadostuje, da je  $\dot{\gamma} = V(\gamma(t))$

$$\gamma(t) = (x_1 \dots x_n)$$

$$\dot{x}_1 = V_1(\gamma(t))$$

$$\vdots$$
$$\dot{x}_n = V_n(\gamma(t))$$

Da zdej obravnavati smo primere za  $n=2$ , vendar v  
povečani obliki  $y'(x) = f(x, y(x))$

$\downarrow$   
 $\gamma$

$\downarrow$   
 $\gamma$

neavtonomni sistemi

Enačbi  $y' = f(x, y)$  priredimo sistem oz avtonomno  
vektorsko p-je

$$\dot{x} = 1$$

$$y = f(x, y)$$

2.12

1. uro

Misl: mdeje dehej panembno

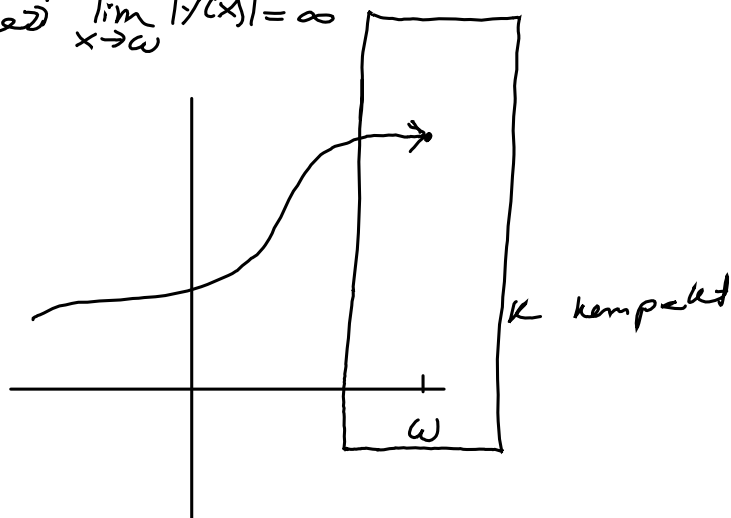
3. naloga  
 $f \in \mathcal{C}^\infty(\mathbb{R})$

$y$  rešitev

$$y' = f(x, y)$$

$y(x_0) = y_0$ , tako da se ne razširi čez  
 $(\alpha, \omega)$   $\omega < \infty$

Dokazi  $\lim_{x \rightarrow \omega} |y(x)| = \infty$



Graf rešitve je veletarsko polje  $V(1, f(x, y))$ . Izberemo kompaktni pravokotnik  $[\omega - \delta, \omega + \delta] \times [-l, l]$

Izrek nam pove: če je  $x$  dovolj blizu  $\omega$ , je toka nica  
bodiš; pod  $-l$  in nad  $l$

Sklep: če  $x$  dovolj blizu  $\omega$  je  $|y(x)| > l$  za  $\forall l$   
 $\Rightarrow \lim_{x \rightarrow \omega} |y(x)| = \infty$

$$y' = \frac{1}{x^2 + y^2}$$

$$y(0) > 0$$

$y$  je definirana na  $\mathbb{R}$

Odvod je omejen, ker je  $y' > 0 \forall x \in \mathbb{R}$ , torej je  $y$  naraščajoča, večji kot je  $x^2$ , večji je  $y$ , vedno manjši je odvod.  $\Rightarrow$

$$y' = \frac{1}{x^2 + y^2} \leq \frac{1}{y^2} \leq \frac{1}{y_0^2} \quad \text{za } x \geq 0$$

$$\leadsto y(x) \leq \frac{1}{y_0}x + y_0 \Rightarrow \text{če je } \omega \text{ končen,}$$

$y(x)$  ne zapusti vsakega kompakta

za  $x < 0$ .  $y$  vedno narašča

$$y' = \frac{1}{x^2 + y^2} \leq \frac{1}{x^2} \leadsto 0 \Rightarrow y' \leadsto 0$$

$\Rightarrow y'$  je omejen

N4.

a) Izrazi polje  $(1,0)$  in  $(0,1)$ , ki sta padeni v polarnih koordinatah  $(r, \varphi)$

b) Določi tok polja  $V(x,y) = a(x^2+y^2)(y,-x)$ , kjer je  $a \in C^\infty(\mathbb{R})$

a)  $\dot{r} = 1 \Rightarrow r(t) = r_0 + t$   
 $\dot{\varphi} = 0 \Rightarrow \varphi(t) = \varphi_0$

$$\begin{aligned} x &= r \cos \varphi \leadsto x(t) = r_0 \cos \varphi_0 + t \cdot \cos \varphi_0 \\ y &= r \sin \varphi \leadsto y(t) = r_0 \sin \varphi_0 + t \sin \varphi_0 \end{aligned}$$

$$\begin{aligned} \dot{x}(t) &= \cos \varphi_0 = \cos \varphi(t) \\ \dot{y} &= \sin \varphi_0 = \sin \varphi(t) \end{aligned}$$

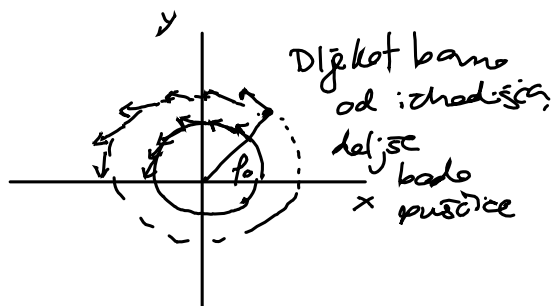
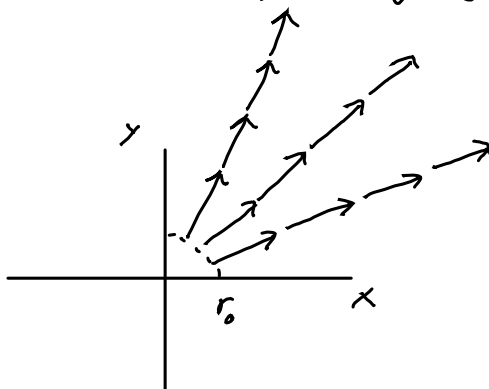
$$\left. \begin{aligned} \dot{x} &= \frac{x}{\sqrt{x^2+y^2}} \\ \dot{y} &= \frac{y}{\sqrt{x^2+y^2}} \end{aligned} \right\} W(x,y)$$

$$\begin{aligned} \dot{r} &= 0 \leadsto r = r_0 \\ \dot{\varphi} &= 1 \leadsto \varphi = \varphi_0 + t \end{aligned}$$

$$\begin{aligned} x &= r \cos \varphi = r_0 \cos(\varphi_0 + t) \\ y &= r \sin \varphi = r_0 \sin(\varphi_0 + t) \end{aligned}$$

$$\dot{x} = -r_0 \sin(\varphi_0 + t) = -y$$

$$\dot{y} = r_0 \cos(\varphi_0 + t) = x$$



$$Q(x,y) = (-y, x)$$

b)

$$a(x^2+y^2) \cdot (y, -x) = a(r^2)(-(-y), x) = ar^2(-Q)$$

↑  
v polarnih  
koordinatah

$$\dot{r} = 0$$

$$\dot{r} = a(r^2) \cdot 0 = 0 \quad \leadsto r = r_0$$

$$\dot{\varphi} = a(r^2) \cdot 1 = a(r^2) \quad \leadsto \varphi(t) = \int a(r^2) dt$$

$$(x, y) = F(r, \varphi)$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J_F \begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix} = J_F \cdot P(r, \varphi) = \varphi_0 + t a(r^2)$$

↑  
mekapdže

$$x(t) = r(t) \cos \varphi(t) = r_0 \cos(\varphi_0 - a(r_0^2)t)$$

$$y(t) = r(t) \sin \varphi(t) = r_0 \sin(\varphi_0 - a(r_0^2)t)$$

$$x(t) = \underbrace{r_0 \cos \varphi_0}_{x_0} \cos(a(r_0^2)t) + \underbrace{r_0 \sin \varphi_0}_{y_0} \sin(a(r_0^2)t) =$$

$$= x_0 \cos(a(x_0^2 + y_0^2)t) + y_0 \sin(a(x_0^2 + y_0^2)t)$$


---

DN  $\dot{r} = \sin r$

$\dot{\varphi} = \cos r$

$$\vec{X} = A(t) \times (t)$$

$$A(t) = \text{konst}$$

$$\hookrightarrow e^{A \cdot t}$$

$$e^A = 1 + A + \frac{A^2}{2!} + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

Gegeben  $A \in \mathbb{R}^{2 \times 2}$

a)  $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

b)  $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad \lambda \in \mathbb{R}$

c)  $A = \begin{bmatrix} a-b & \\ b & a \end{bmatrix}$

a) 
$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} (At)^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{t^k}{k!} & 0 \\ 0 & \frac{(2t)^k}{k!} \end{bmatrix} =$$

$$A^k = \begin{bmatrix} 1 & 0 \\ 0 & 2^k \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$

b) Jordane'sche Normalform

$$A = \lambda I + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

~~$A^k = (\lambda I + N)^k$~~

$$e^{At} = e^{(\lambda I + N)t} \xleftarrow{\text{kommutativ}} e^{\lambda I t} e^{Nt} =$$

linear unabhängig  
id.  $A + \frac{1^2}{2} + \dots$

$$= e^{\lambda t} \left( \sum_{k=0}^{\infty} \frac{t^k}{k!} N^k \right) =$$

$$= e^{\lambda t} \left( \frac{t^0}{0!} I + \frac{t^1}{1!} N + \frac{t^2}{2!} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) =$$

$$e^{\lambda t} \begin{bmatrix} 1 + \frac{t^2}{2} & & \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

c)  $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$  Ugotovimo: to je matrika s kompleksnimi

Ans  $a+bi$ :

$$e^{At} = e^{(a+bi)t} = e^{at} e^{ibt} = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

$$A = aI + bJ$$

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$J^2 = -I$$

$$J^3 = -J \quad J^4 = I$$

$$e^{At} = e^{(aI+bJ)t} = e^{bt} e^{bJt} =$$

$$= e^{bt} \left( I + bJ \frac{t}{1} + \frac{b^2 J^2 t^2}{2!} + \frac{b^3 J^3 t^3}{3!} + \frac{b^4 J^4 t^4}{4!} + \frac{b^5 J^5 t^5}{5!} + \dots \right) =$$

$$= e^{bt} \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (bt)^{2k} I + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (bt)^{2k+1} J \right) =$$

$$= e^{bt} (\cos bt I + \sin bt J) = e^{bt} \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}$$

## Sistemi NDE s konstantnimi koeficienti

$$\dot{\vec{x}} = A\vec{x} \quad A \in M_{n \times n}$$

Vseke rešitev je oblike  $\vec{x}(t) = \bar{\Phi}(t)\vec{x}(0)$ , kjer je

$\bar{\Phi}$  fundamentalna rešitev (reš. matrika) sistema

$$\dot{\bar{\Phi}} = A\bar{\Phi} \quad \bar{\Phi}(0) = id$$

Kako poiščemo  $\bar{\Phi}$ ?

$$\begin{aligned} \psi &= e^{At} & \psi(0) &= I \\ \dot{\psi} &= Ae^{At} = A\psi(t) \end{aligned} \quad \checkmark$$

$$\bar{\Phi} = \psi$$

$A = PJP \dots$  jordanova forma

$$e^{At} = e^{(PJP^{-1})t} = \sum \frac{P J^k P^{-1} t^k}{k!} = P e^{Jt} P^{-1}$$

Predpostavljajmo, da ima  $A$  realne lastne  
vrednosti

Paisi fundametalno resiku sistema

$$\dot{X} = X - Y + 4Z$$

...

we febo!

Pózišni Änderungen ~ 5

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 3 & 3 & 4 \\ -2 & -1 & -1 \end{bmatrix} \quad \begin{bmatrix} 3-\lambda & 1 & 2 \\ 3 & 3-\lambda & 4 \\ -2 & -1 & -1-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) =$$

$$(3-\lambda)(\lambda^2 + \lambda(1-3) + 4) - 1(-3\lambda - 3 + 8) + 2(-3 + 6 - 2\lambda)$$

$$= (3-\lambda)(\lambda-1)^2 + (-\lambda+1) =$$

$$= (\lambda-1)(-\lambda^2 + \lambda(3+1) - 3 - 1) =$$

$$= -(\lambda-1)(\lambda^2 - 4\lambda + 4) = -(\lambda-1)(\lambda-2)^2$$

$$J = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix} \quad P = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 1:$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 4 \\ -2 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & -2 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

$$x=0$$

$$y=-2z$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 2$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 4 \\ -2 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\ker(A-2I):$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$x+z=0$$

$$\ker(A-2I) = \text{Lin} \left\{ \begin{bmatrix} x \\ y \\ -x \end{bmatrix} \right\} =$$

$$\text{Lin} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

~ je v jedné, a i k další vlně

$$\vec{v}_1 = (A-2I)\vec{v}_2 \in$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad J = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix}$$

$$e^{Jt} = \begin{bmatrix} e^t & & \\ & e^{2t} & \\ & & e^{2t} \end{bmatrix}$$

Nehme mögen: lineares System NDE  
 konst. Koeff.

$$\dot{\vec{v}} = A\vec{v} + f(t) \quad // \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$\vec{v}_h = e^{At} \vec{v}_0 \quad \vec{v}_p = e^{At} \cdot w(t)$$

$$\vec{v}_p = \underbrace{A e^{At} w}_{\parallel A \vec{v}_p} + e^{At} \cdot \dot{w} = A \vec{v}_p + f$$

$$e^{At} \cdot \dot{w} = f$$

$$\dot{w} = e^{-At} \cdot f$$

$\vec{\Phi}(t)$  Fundamentalsystem  $\Rightarrow \vec{\Phi}$  invertierbar  
 vse  $t$ , hier je definiert  
 $\vec{\Phi}(t) = \vec{\Phi}(-t)$

$$w(t) = \int_0^t e^{-A\xi} f(\xi) d\xi$$

$$v_p = e^{At} \cdot w(t) = \int_0^t e^{A(t-\xi)} f(\xi) d\xi$$

Poisk: splošna rešitev sistema

$$\dot{x} = x - y + \frac{1}{\cos t}$$

$$\dot{y} = 2x - y$$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = -(1-\lambda)(1+\lambda) + 2 = \lambda^2 - 1 + 2 = \lambda^2 + 1$$

$$\lambda = \pm i$$

uvodi nove koordinate  $u = x$

$$v = y - x$$

$$\dot{v} = \dot{y} - \dot{x}$$

$$y = v + u$$

$$\dot{y} = \dot{v} + \dot{u}$$

$$\dot{u} = u - v - u + \frac{1}{\cos t} = -v + \frac{1}{\cos t}$$

$$\dot{v} - \cancel{\frac{1}{\cos t}} = 2u - \cancel{\frac{1}{\cos t}} - u =$$

$$\dot{v} = u + \frac{1}{\cos t}$$

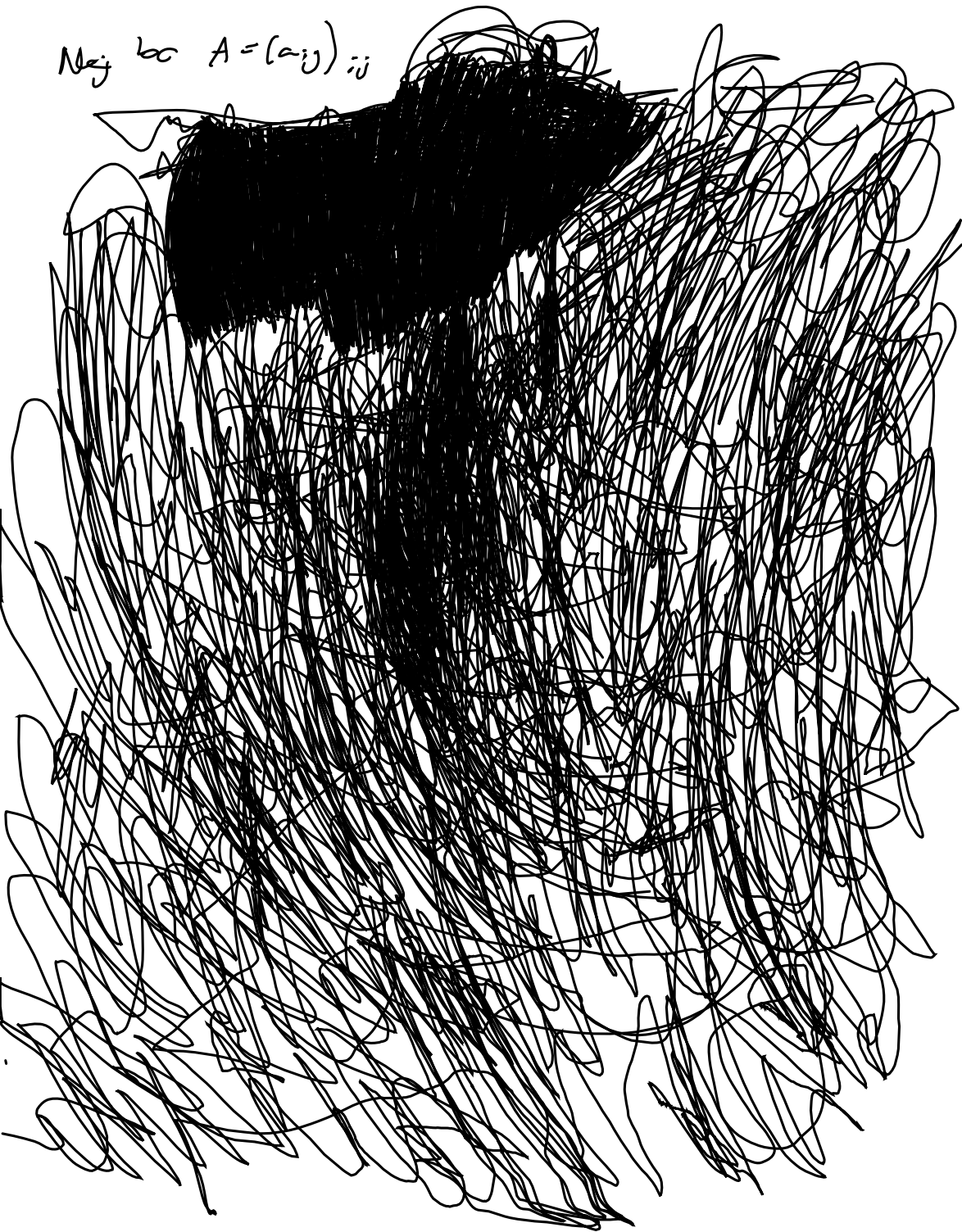
$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad e^{Bt} = e^{it} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$w = \int_a^t e^{-i\zeta} \begin{bmatrix} \frac{1}{\cos \zeta} \\ -\frac{1}{\cos \zeta} \end{bmatrix} d\zeta = e^{-Bt} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

$$w = \begin{bmatrix} \int_a^t (1 - \tan \zeta) d\zeta \\ \int_a^t (-\tan \zeta - 1) d\zeta \end{bmatrix} = \begin{bmatrix} t + \ln |\cos t| \\ -t - \ln |\cos t| \end{bmatrix}$$

$$V(t) = e^{Bt} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + e^{Bt} w(t)$$

Nej bo  $A = (a_{ij})_{ij}$



no steady

$$x_j(t) = e^{\lambda_1 \dots t} \cdot p_i(t)$$

$st(p_i) \leq \text{alg.}$   
red. late values  
as  $\lambda_i$

Vide jo kuh

12.12

Spomnimo se

$$\dot{\vec{x}} = A \cdot \vec{x} + f(z)$$

$$\Phi = P e^{J^+} Q^{-1}$$

$$x_n = \Phi x_0$$

$$x_r = \int_a^+ \Phi(t) \Phi(s)^{-1} f(s) ds$$

N: Določiti splošno rešitev sistema

$$\dot{x} = y - z + t$$

$$\dot{y} = x - z + 1$$

$$\dot{z} = 2x + 2y - 3z + t + 1$$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 2 & 2 & -3 \end{bmatrix}$$

$$f(t) = \begin{bmatrix} t \\ 1 \\ t+1 \end{bmatrix}$$

$$\det(A - \lambda I) =$$

$$\begin{vmatrix} -\lambda & 1 & -1 \\ 1 & -\lambda & -1 \\ 2 & 2 & -3-\lambda \end{vmatrix} = -\lambda(3\lambda + \lambda^2 + 2) - 1(-3-\lambda+2) - 1(2+2\lambda)$$

$$= -\lambda((\lambda+1)(\lambda+2)) + (\lambda+1) - 2(\lambda+1) =$$

$$= (\lambda+1)(\lambda^2 + 2\lambda - 1) = \dots \text{ kps} = -(\lambda+1)^3$$

$$A+I = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \ker(A+I) = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} = \text{Lin} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\ker(A+I)^2 = \mathbb{R}^{3 \times 3}$$

(ker se dimenzija jedra poveča za vsaj 1)

$$\Rightarrow \text{dopolnimo do baze: } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

drugi stolpec

$$(A+I) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \in \ker(A+I)$$

↑ tretji stolpec

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{ali pa} \quad P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

oba sta vredni

(ker sta oba lin. neod. od  $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ )

$$P^{-1} = \left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -2 \end{array} \right]$$

$$\sim \dots \quad P^{-1} = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad e^{Jt} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & -te^{-t} \\ 0 & 0 & e^{-t} \end{bmatrix}$$

$$\Phi(t) = e^{At} = P e^{Jt} P^{-1}$$

$$\Phi^{-1}(s) = P e^{-Js} P^{-1}$$

$$x_p = P e^{Jt} P^{-1} \int_0^t P e^{-Js} P^{-1} f(s) ds =$$

$$= P e^{Jt} \int_0^t e^s \begin{bmatrix} 1 & 1 & -s \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s \\ 1 \\ s+1 \end{bmatrix} ds$$

$$= P e^{Jt} \int_0^t e^s \begin{bmatrix} 1 & 1 & -s \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s-1 \\ 1 \\ 0 \end{bmatrix} ds =$$

$$= P e^{Jt} \int_0^t e^s \begin{bmatrix} s-1 \\ 1 \\ 0 \end{bmatrix} ds = P e^{Jt} \begin{bmatrix} se^s - 2e^s \\ e^s \\ 0 \end{bmatrix} \Big|_0^t =$$

$$\int s e^s ds = s e^s - \int e^s ds = s e^s - e^s$$

$$u = s \quad du = e^s$$

$$du = 1 \quad v = e^s$$

$$= P e^{Jt} \left( e^t \begin{bmatrix} t-2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right) = x_p(t)$$

splošna rešitev:

$$x(t) = \Phi(t) x_0 + x_p(t)$$

## Linearne enačbe reda $n$ z konstantnimi koeficienti

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$$

Sistem

$$x_1 = y$$

$$x_2 = y'$$

$$\vdots$$

$$x_n = y^{(n-1)}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ -\frac{a_1}{a_0} x_{n-1} - \frac{a_2}{a_0} x_{n-2} + \dots + \frac{a_n}{a_0} x_1 \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Dobimo sistem oblike  $\vec{\dot{v}} = A\vec{v}$

Komponente rešitev so oblike:  $\sum p(t) e^{\lambda t}$

Splošna rešitev:  $y_s = \sum_{\lambda_i} p_i(x) e^{\lambda_i x}$

Če je  $\lambda$  večkratna ničla je  $y = p(x) e^{\lambda x}$ , kjer je  $p$  polinom iste stopnje kot je večkratnost ničle

a)

$$y^{(n)} - 4y = 0$$

$$y = e^{\lambda x}$$

$$\lambda^n e^{\lambda x} - 4e^{\lambda x} = 0$$

$$\lambda^n - 4 = 0$$

$$n=4$$

$$(\lambda^2 - 2)(\lambda^2 + 2) = (\lambda - \sqrt{2})(\lambda + \sqrt{2})(\lambda - \sqrt{2}i)(\lambda + \sqrt{2}i)$$

$$y_s = a_1 e^{\sqrt{2}x} + a_2 e^{-\sqrt{2}x} + a_3 e^{\sqrt{2}ix} + a_4 e^{-\sqrt{2}ix}$$

$$= a_1 e^{\sqrt{2}x} + a_2 e^{-\sqrt{2}x} + \underbrace{\cos(\sqrt{2}x)}_{\text{eine Konstante}} (a_3 + a_4) + \underbrace{\sin(\sqrt{2}x)}_{\text{eine Konstante}} (a_3 - a_4)$$

b)  $y''' + y'' + y' + y = 0$

$$\lambda^3 + \lambda^2 + \lambda + 1 = 0$$

$$\lambda^2(\lambda + 1) + (\lambda + 1) = (\lambda + 1)(\lambda - i)(\lambda - i) = 0$$

$$y_s = a_1 e^{-x} + a_2 \cos x + a_3 \sin x$$

$$c) y^{(4)} - 8y^{(3)} + 18y'' - 27y' = 0$$

$$\lambda^4 - 8\lambda^3 + 18\lambda^2 - 27\lambda = 0$$

$$\begin{array}{r|rrrrr} -1 & 1 & -8 & +18 & 0 & -27 \\ & \downarrow & & & & \\ & 1 & -9 & 27 & -27 & 0 \end{array}$$

$$\begin{array}{r|rrrr} 3 & 1 & -9 & 27 & -27 & 0 \\ & \downarrow & & & & \\ & 1 & -6 & 9 & 0 & 0 \end{array}$$

$$\begin{array}{r|rrrr} +3 & 1 & -6 & 9 & 0 & 0 \\ & \downarrow & & & & \\ & 1 & -3 & 0 & 0 & 0 \end{array}$$

$$= (\lambda + 1)(\lambda + 3)^3$$

$$y_s = C e^{-x} + (C_2 x^2 + C_3 x + C_4) e^{3x}$$

$$a_n x^n y^{(n)} + \dots a_0 y = 0$$

Take  $x = e^t$  gives ELP

$$a_n e^t y^{(n)} + \dots a_0 y =$$

$$z(t) = y(e^t)$$

$$z'(t) = e^t (y'(e^t))$$

$$z''(t) = e^t y'(e^t) + e^{2t} y''(e^t)$$

$$z'''(t) = e^t y'(e^t) + y''(e^t)(e^{2t} + e^{2t}) + e^{3t} y'''(e^t)$$

$$y'(e^t) = \frac{z'(t)}{e^t}$$

$$y''(e^t) = \frac{z''(t) - e^t y'(e^t)}{e^{2t}} = \frac{z''(t) - z'(t)}{e^{2t}}$$

$$y'''(e^t) = \frac{-z'(t) + 2z''(t) + z'''(t)}{e^{3t}}$$

$\underbrace{\hspace{10em}}_{L(z''(t))}$

$$z^{(k)}(t) = x^k y^{(k)} + L(z' \dots z^{(k-1)})$$

$$z^{(k+1)} = k x^k y^{(k)} + x^k y^{(k+1)} + L(z' \dots z^{(k)})$$

$$z^{(k+1)} = k e^{kt} y^{(k)} +$$

Dobiti splošno rešitev

a)  $x^3 y''' - 2xy' + 4y = 0$

b)  $x^4 y^{(4)} + 6x^3 y''' + 3xy' + y = 0$

a)  $y = x^\lambda$

$$y' = \lambda x^{\lambda-1} \quad y'' = \lambda(\lambda-1)x^{\lambda-2} \quad y''' = \lambda(\lambda-1)(\lambda-2)x^{\lambda-3}$$

$$x^3 \cdot \lambda(\lambda-1)(\lambda-2)x^{\lambda-3} - 2x \cdot \lambda x^{\lambda-1} + 4x^\lambda$$

$$(\lambda(\lambda-1)(\lambda-2) - 2\lambda + 4)x^\lambda = 0$$

$$(\lambda-2)(\lambda^2 - \lambda - 2) = (\lambda-2)(\lambda-2)(\lambda+1) = (\lambda-2)^2(\lambda+1)$$

$$y_1 = C_1 \frac{1}{x} \quad y_2 = (C_2 \ln x + C_3)x^2$$

b)  $x^\lambda (\lambda(\lambda-1)(\lambda-2)(\lambda-3) + 6\lambda(\lambda-1)(\lambda-2) + 3(\lambda-1) + 1) = 0$

$$\lambda(\lambda-1)(\lambda-2)(\lambda-3+6) + \underbrace{3\lambda-3+1}_{3\lambda-2}$$

$$\lambda(\lambda^2 - 3\lambda + 2)(\lambda+3) + 3\lambda - 2 =$$

$$= \lambda(\lambda^3 + \lambda^2(3-3) + \lambda(2-9) + 6 + 3\lambda - 2) =$$

$$= \lambda^4 - 7\lambda^2 + 3\lambda + 4$$

Ups

$$\lambda^4 + 2\lambda^2 + 1 = (\lambda^2 + 1)^2 = (\lambda - i)^2(\lambda + i)^2$$

$$(C_1 \ln x + C_2)x^i + (C_2 \ln(x) + C_3)x^{-i}$$

$$\underset{e^{i \ln x}}{e^{i \ln x}}$$

$$e^{i \ln x} = \cos(\ln x) + i \sin(\ln x)$$

$$\cos(\ln x)(C_1 \ln x + C_2 + C_3 \ln x + C_4) + i \sin(\ln x)(C_1 \ln x + C_2 - C_3 \ln x - C_4) =$$

$$= \cos(\ln x)(\ln x(C_1 + C_3) + C_2 + C_4) + i \sin(\ln x)(\ln x(C_1 - C_3) + C_2 - C_4) =$$

$$\underbrace{\hspace{10em}}_{i \text{ nebesno število } \vee \text{ konstanta}}$$

Dobimo  $(C_1 \ln x + C_2) \cos(\ln x) + (C_3 \ln x + C_4) \sin(\ln x)$

$$a_n y^{(n)} + \dots + a_1 y' + a_0 y = e^{\mu x} p(x)$$

če je  $y_p$  neka rešitev, je vsaka druga rešitev oblike

$$y_h + y_p$$

$y_p$  iščemo z nastavkom  $e^{\mu x} q(x) x^k$

$q \dots$  polinom iste stopnje kot  $p$

$k \dots$  vrednost  $\mu$  v  $P(\lambda)$

karakteristion: polinom homogenega dela

Pois. spl. & ee.

$$a) y'' - 5y' + 4y = 4x^2 e^{2x}$$

$$H: y'' - 5y' + 4y = 0$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$(\lambda - 1)(\lambda - 4) = 0$$

$$y_H = C_1 e^x + C_2 e^{4x}$$

$$P: y_P = e^{2x} (Ax^2 + Bx + C)$$

$$\begin{aligned} y_P' &= 2e^{2x} (Ax^2 + Bx + C) + e^{2x} (2Ax + B) = \\ &= e^{2x} (x^2(2A) + x(2B + 2A) + (2C + B)) \end{aligned}$$

$$\begin{aligned} y_P'' &= 2e^{2x} (x^2(2A) + x(2B + 2A) + 2C + B) + \\ &+ e^{2x} (4xA + 2B + 2A) \end{aligned}$$

$$x^2: 4A + 2A + A = 1 \Rightarrow A = -2$$

$$x: 4A + 2B + 2A + 2B + 2A - B = -5 \Rightarrow B = 2$$

$$1: 2A + 2B + 2B + 4C - 5 \cdot 2C - 5B + 4C \Rightarrow C = -3$$

$$y_S = y_H + y_P = C_1 e^x + C_2 e^{4x} + e^{2x} (-2x^2 + 2x - 3)$$

$$y'' + 3y' - 4y = e^{-4x} + x e^{-x}$$

$$H: \lambda^2 + 3\lambda - 4 = 0$$

$$(\lambda + 4)(\lambda - 1) = 0$$

$$y = C_1 e^{-4x} + C_2 e^x$$

$$1. \quad -||- = e^{-4x}$$

$$y_p = y_1 + y_2$$

$$y_1 = e^{-4x} C x$$

$\hookrightarrow$  ker je -4 nide pdrnane

$$y_1' = -4e^{-4x} C x + e^{-4x} C$$

$$y_1'' = 16e^{-4x} C x - 8e^{-4x} C$$

$$x^2: = 0$$

$$x: 16C + 3(-4)C - 4 \cdot C = 0$$

$$1: -8C + 3C - 4C = 1$$

$$C = -\frac{1}{85}$$

$$2. \quad -||- = x e^{-x}$$

$$y_2 = e^{-x} (Ax + B)$$

$$y_2' = -e^{-x} (Ax + B) + e^{-x} A$$

$$y_2'' = e^{-x} (Ax + B) - A e^{-x} + e^{-x} A$$

$$x: A - 3A - 4A = 1 \Rightarrow A = -\frac{1}{6}$$

$$1: B - 2A = \dots \Rightarrow B = -\frac{1}{36}$$

$$y = C_1 e^{-4x} + C_2 e^x + \frac{1}{5} x e^{-4x} - \frac{1}{6} e^{-x} \left(x + \frac{1}{6}\right)$$

$$c) y'' + 2y' + 2y = e^{-x} \cos x$$

$$H: y'' + 2y' + 2y = 0$$

$$\lambda^2 + 2\lambda + 2 = 0$$

$$D = 4 - 8 = -4$$

$$\lambda_{1,2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$P: y_h = e^{-x} (c_1 \cos x + c_2 \sin x)$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$y'' + 2y' + 2y = \frac{1}{2} (e^{x(i-1)} + e^{x(-1-i)})$$

$$y_1 = e^{x(i-1)} A x$$

$$y_1' = (i-1) e^{x(i-1)} A x + A e^{x(i-1)}$$

$$y_1'' = A(i-1) e^{x(i-1)} + (i-1) e^{x(i-1)} A + A(i-1)^2 e^{x(i-1)} x$$

$$x: \dots = 0 \Rightarrow A = \frac{1}{2}$$

$$1: \dots = \frac{1}{2} \Rightarrow A = \frac{1}{2}$$

$$y_2:$$

$$a_n x^n y^n + \dots + a_1 y = x^\mu p(\ln x) \quad \text{Eulerjeva DE}$$

rešavek:  $x^\mu g(\ln x) (\ln x)^k \leftarrow$  stopnja  $\mu$  v karakterističnem polinomu  
 $\uparrow$  iste stopnje kot  $p$

$$\# : x^2 \lambda(\lambda-1) x^{\lambda-2} - 4x \lambda x^{\lambda-1} + 6x^{\lambda} = 0$$

$$\lambda(\lambda-1) - 4\lambda + 6 = 0 \quad \uparrow y = x^{\lambda}$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda-2)(\lambda-3) = 0$$

$$y_h = C_1 x^2 + C_2 x^3$$

$$P: \quad \swarrow x^\mu \quad \swarrow g(\ln x)$$

$$y_p = x(A \ln x + B)$$

$$y_p' = A \ln x + B + x \left( \frac{A}{x} \right) = A \ln x + A + B$$

$$y_p'' = \frac{A}{x}$$

$$x^2 \left( A \frac{1}{x} \right) - 4x(A \ln x + A + B) + 6x(A \ln x + B) = 2x \ln x$$

$$(A + 6B + 4A + 4B)x + x \ln x (4A + 6A) = 2x \ln x$$

$$10B + 5A = 0$$

$$10A = 2$$

$$10B = -1$$

$$A = \frac{1}{5}$$

$$B = -\frac{1}{10}$$

$$Vf3: \Rightarrow A = 1$$

$$B = \frac{3}{2}$$

$$y_s = y_h + y_p = C_1 x^2 + C_2 x^3 + x \left( \ln x + \frac{3}{2} \right)$$

Dokaži da je  $z = t = \arcsin x$

$(1-x^2)y'' - xy' - 4y = 0$  prevede se homogena  
skupšt. koef. <sup>2 PE</sup>

$$z(t) = y(\sin t) \quad x(t) = \sin t$$

$$z'(t) = \cos t y'(\sin t)$$

$$z''(t) = \cos^2 t y''(\sin t) - \sin t y'(\sin t)$$

$$\underbrace{\cos^2 t y''}_{z''(t)} - \underbrace{\sin t y'}_{4z(t)} = 0$$

$$z''(t) - 4z(t) = 0$$

$$z = e^{\lambda t}$$

$$\lambda^2 - 4 = (\lambda - 2)(\lambda + 2)$$

$$z_1 = C_1 e^{2t} = C_1 e^{2 \arcsin x} = y_1(x)$$

$$z_2 = C_2 e^{-2t} = C_2 e^{-2 \arcsin(x)} = y_2(x)$$

## Variacijs konstante

$$a_0 y^n + \dots + a_n y = f$$

$$\Phi(x) = \begin{bmatrix} y_1 & \dots & y_n \\ y_1' & & y_n' \\ \vdots & & \vdots \\ y_1^{(n)} & & y_n^{(n)} \end{bmatrix}$$

$$\Phi \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \leadsto y_p = \Phi \begin{bmatrix} c_1(x) \\ \vdots \\ c_n(x) \end{bmatrix}$$

$$y_p' = \Phi' \cdot C + \Phi C'$$

$$A = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 1 & \\ -\frac{a_n}{a_0} & & 0 & & -\frac{a_1}{a_0} \end{bmatrix}$$

$$p = \begin{cases} p_1 = y \\ p_2 = y' \\ \vdots \\ p_n = y^{(n-1)} \end{cases}$$

$$\dot{p} = A p + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ f \end{bmatrix}}_{\equiv \vec{z}}$$

$$\dot{\Phi} = A \Phi \quad A \Phi C$$

$$y_p' = \Phi' C + \Phi C' = A \Phi C + \vec{z}$$

$$\text{Dobivamo: } \Phi C' = \vec{z}$$

Skladj  $C = C(x)$  dobivamo iz ove konst:

$$\begin{bmatrix} y_1 & \dots & y_n \\ \vdots & & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1' \\ \vdots \\ c_n' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ f/a_0 \end{bmatrix}$$

Doloci splošno rešitev DE

$$y'' + 3y' + 2y = \ln(1+e^x)$$

homogeno:

$$y'' + 3y' + 2y = 0$$

nastavek  $y = e^{\lambda x}$

$$y' = \lambda e^{\lambda x}$$

$$y'' = \lambda^2 e^{\lambda x}$$

$$\lambda^2 + 3\lambda + 2 = 0 \quad \lambda_1 = -1$$

$$(\lambda + 1)(\lambda + 2) = 0 \quad \lambda_2 = -2$$

$$y = C_1 e^{-x} + C_2 e^{-2x}$$

$$y_1 = e^{-x} \quad \dot{y}_1 = -e^{-x}$$

$$y_2 = e^{-2x} \quad \dot{y}_2 = -2e^{-2x}$$

$$\Phi = \begin{bmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{bmatrix}$$

$$\Phi \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \ln(1+e^x) \end{bmatrix}$$

$$e^{-x} C_1'(x) + e^{-2x} C_2'(x) = 0$$

$$-e^{-x} C_1'(x) - 2C_2'(x)e^{-2x} = \ln(1+e^x) \quad \left. \vphantom{\frac{1}{2}} \right\} +$$

$$-C_2' e^{-2x} = \ln(1+e^x)$$

$$C_2' = -e^{2x} \ln(1+e^x)$$

$$C_2 = \int -e^{2x} \ln(1+e^x) dx = \int e^u (e^u - 1) u dx$$

$$e^u = 1+e^x$$

$$e^u du = e^x dx$$

$$C_2 = \int e^{2u} u - e^u u du = u e^u - e^u - \frac{u}{2} e^{2u} + \frac{1}{4} e^{2u}$$

$$u = u \quad dv = e^u du$$

$$du = du \quad v = e^u$$

$$C_2(x) = (1+e^x) \ln(1+e^x) - (1+e^x) - \frac{\ln(1+e^x)}{2} (1+e^x)^2 + \frac{1}{4} (1+e^x)^2$$

$$\ln(1+e^x) \left( 1+e^x - \frac{(1+e^x)^2}{2} \right) - 1 - e^x + \frac{1}{4} (1+2e^x+e^{2x}) =$$

$$= \ln(1+e^x) \left( \frac{1}{2} (1-e^{2x}) \right) - \frac{3}{4} - \frac{1}{2} e^x + \frac{1}{4} e^{2x}$$

$\leftarrow$  kerje konstanta  $C$

$$C_1' = -C_2' e^{-x} = e^x \ln(1+e^x)$$

$$u = \ln(1+e^x)$$

$$C_1 = \int u e^u - e^u = \ln(1+e^x) (1+e^x) - 1 - e^x =$$

$$(1+e^x) (\ln(1+e^x) - 1)$$

$$y_3 = C_1 y_1 + C_2 y_2 + y_h =$$

$$(e^{-x} + 1) (\ln(1+e^x) - 1) + \frac{1}{4} - \frac{1}{2} e^{-x} + \ln(1+e^x) \left( \frac{e^{-2x} - 1}{2} \right)$$

~~701~~

$$+ a e^{-x} + b e^{-2x}$$

## Determinante Wronskiege

$$y_1, \dots, y_n \in \mathcal{C}^{n-1}([a, b])$$

$$W = \det \begin{bmatrix} y_1 & \dots & y_n \\ y_1' & \dots & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix}$$

V primera ho sta  $u, v$  lin. neodvisn. rešitvi:

$$y'' + p(x)y + q(x) = 0$$

$$\leadsto W = u'v - uv'$$

iz Liouville formule sledi, da je  $W' + p(x)W = 0$

$$\leadsto \ln W = -\int p dt$$

$$W = W_0 e^{-\int p dt}$$

Podana je DE

$$(1+x^2)y'' = 2y$$

a)  $y_1 = 1+x^2$

rešitev?

b) Dalo bi splošno rešitev

a)  $(1+x^2) \cdot 2 = 2(1+x^2)$  ✓

b) Naj bo  $y_2$  lin. neod. od  $y_1$  rešimo enačbo

$$W = y_1 y_2' - y_2 y_1' \neq 0$$

$(1+x^2)y'' - 0 \cdot y' - \frac{2}{1+x^2}y = 0$

$$W' + pW = 0 \rightarrow \text{ker}$$

$$W' = 0$$

$$W = C = 1 \leftarrow \text{izvolimo}$$

$$1 = y_1 y_2' - y_2 y_1' = y_2'(x^2+1) - y_2(2x)$$

$$\text{Hi: } y_2' = y_2 \frac{2x}{x^2+1}$$

$$\ln y_2 = \int \frac{2x}{x^2+1} dx = \ln(1+x^2) + C$$

$$y_2 = C(1+x^2)$$

$$P: C'(1+x^2)^2 = 1 \Rightarrow C = \int \frac{1}{(1+x^2)^2} dx$$

$$C = A \arctan x + B \ln(1+x^2) + \frac{Dx+E}{1+x^2}$$

↓

$$C' = \frac{A}{1+x^2} + \frac{2Bx}{1+x^2} + \frac{D(1+x^2) - 2x(Dx+E)}{(1+x^2)^2}$$

$$= \frac{A(1+x^2) + 2B(x+x^3) + D(1+x^2) - 2Dx^2 - 2DEX}{(1+x^2)^2}$$

$$x^3: 2B = 0$$

$$x^2: A + D - 2D = A - D = 0 \Rightarrow A = D$$

$$x: 2B - 2E = 0 \Rightarrow E = 0$$

$$1: A + D = 1 \Rightarrow A = D = \frac{1}{2}$$

$$C = \frac{1}{2} \arctan x + \frac{\frac{1}{2}x}{1+x^2} + D$$

$$y_p = C(1+x^2) = D(1+x^2) + \frac{1}{2}(1+x^2) \arctan x + \frac{1}{2}x$$

$$y_2 = (1+x^2) \arctan x + x$$

leško množimo z 2, ker je itak konstanta spredaj

Naj bo  $g \in L_1(\mathbb{R})$  in naj bo  $y \in C^2(\mathbb{R})$  rešitev DE

$$y'' + g y = 0$$

a) Dokaži  $\lim_{x \rightarrow \infty} y'(x) = 0$

b) Dokaži da  $\exists$  neomejen rešitev DE

$$a) \quad y'(x) = \int_a^x y''(t) dt = \int_a^x -g y(t) dt + y'(a)$$

$$|y'(x)| \leq \int_a^x |g y(t)| dt + |y'(a)| \quad |y| \leq M$$

$$\leq M \int_a^x |g(t)| dt + |y'(a)| < \infty \text{ za}$$

Dokazati bomo da je odvod od nekega  
repres monotona  $\Rightarrow$  gre proti 0

$$\lim_{x \rightarrow \infty} y'(x) = \int_a^\infty -g y(t) dt + y'(a) = 0 \text{ ker je } y \text{ omejen}$$

b)  $\exists$  lin. neodvisna rešitvi  $z$

$$\begin{vmatrix} y & z \\ y' & z' \end{vmatrix} = yz' - zy' \neq 0$$

$$w' + p w = 0 \Rightarrow w' = 0 \\ \Rightarrow w' = c \\ \text{reimo 1}$$

$$z(\infty) = y(\infty) z' - z(\infty) \underbrace{y'(\infty)}_0$$

$$y(\infty) z'(\infty) = 1$$

$$z'(\infty) \neq 0 \Rightarrow z \text{ neomejena}$$

$$(1+x^2) y'' + xy' - n^2 y = 0$$

$$a) y_1 = (x + \sqrt{1+x^2})^n \text{ je rešitev}$$

$$y_1' = \left(1 + \frac{x}{\sqrt{1+x^2}}\right) n (x + \sqrt{1+x^2})^{n-1}$$

$$y_1'' = \left(\frac{1 - \frac{x}{\sqrt{1+x^2}}}{1+x^2}\right) n (x + \sqrt{1+x^2})^{n-1} + n \left(1 + \frac{x}{\sqrt{1+x^2}}\right) (n-1) \left(1 + \frac{x}{\sqrt{1+x^2}}\right)^{n-2}$$

$\uparrow$   
za  $n \geq 2$

$$\text{za } n=1: y_1' = 1 + \frac{x}{\sqrt{1+x^2}} \quad y_1'' = \frac{1 - \frac{x}{\sqrt{1+x^2}}}{1+x^2}$$

$$\begin{aligned} (1+x^2) y_1'' &= \left(1 + \frac{x}{\sqrt{1+x^2}}\right)^{n-2} \left( n \frac{-x + \sqrt{1+x^2}}{\sqrt{1+x^2}} \cdot (x + \sqrt{1+x^2}) + n(n-1) \frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}} \right) \\ &= \left( \frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}} \right)^{n-2} \left( \frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}} \right) (n(\sqrt{1+x^2} - x) + n(n-1)) \\ &= n \left( \frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}} \right)^{n-1} (\sqrt{1+x^2} - x + n - 1) \end{aligned}$$

$$x y_1' = \frac{(x + \sqrt{1+x^2})^n}{\sqrt{1+x^2}} n x$$

$$n^2 y_1 = n^2 (x + \sqrt{1+x^2})^n$$

$$(\sqrt{1+x^2} + x)^n \left( \frac{n(n-1) + \frac{n}{\sqrt{1+x^2}} + \frac{nx}{\sqrt{1+x^2}} + n^2}{\dots} \right)$$

$$= y_1 \left( \frac{n(x+1)}{\sqrt{1+x^2}} + 2n^2 - n \right) \quad \text{neleže je napaka}$$

$$y_1'' = n(n-1) \frac{(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}} \left( \frac{1}{(1+x^2)} \right) n (x+\sqrt{1+x^2})^{n-1}$$

$$= \frac{(x+\sqrt{1+x^2})^{n-1}}{\sqrt{1+x^2}} \left( n(n-1)(x+\sqrt{1+x^2}) + \frac{n}{1+x^2} \right) \dots$$

b) durch splines rechte

$$w = \begin{vmatrix} y_1 & y \\ y_1' & y' \end{vmatrix}$$

$$w' + p w = 0$$

$$p = \frac{x}{1+x^2}$$

$$\frac{dw}{w} = -\frac{x}{1+x^2} dx$$

$$\ln w = -\frac{1}{2} \ln(1+x^2) + C$$

$$y_1 y' - y y_1' = \frac{C}{\sqrt{1+x^2}}$$

$$w = \frac{C}{(1+x^2)^{\frac{1}{2}}} = \frac{C}{\sqrt{1+x^2}}$$

$$y' - y \frac{y_1'}{y_1} = \frac{C}{(1+x^2)^{\frac{3}{2}}}$$

$$y_1' = (x+\sqrt{1+x^2})^n n \frac{1}{\sqrt{1+x^2}}$$

$$y' - y n \frac{1}{\sqrt{1+x^2}} = \frac{1}{(1+x^2)^{\frac{3}{2}}} \quad C=1$$

Homogen:

$$y' = y_1$$

mit var konst:

$$y_2 = C(x+\sqrt{1+x^2})^n \frac{1}{(1+x^2)^{\frac{3}{2}}}$$

$$y_2' = C'(x) y_1 + C(x) y_1'$$

$$C'(x) y_1 + C(x) y_1' - \underbrace{C(x) y_1 n \frac{1}{\sqrt{1+x^2}}}_{C(x) y_1'} = \frac{1}{(1+x^2)^{\frac{3}{2}}}$$

$$C'(x) y_1^2 = \frac{1}{(1+x^2)^{\frac{3}{2}}}$$

$$C'(x) = \frac{1}{(1+x^2)^{\frac{3}{2}} (x+\sqrt{1+x^2})^{2n}}$$

$$x = \sinh p$$

$$dx = \cosh p \, dp \quad C = \int \frac{\cosh p}{(\cosh^2)^{\frac{3}{2}} (\sinh + \cosh)^{2n}} =$$

$$= \frac{(\cosh + \sinh)^{2n}}{\dots} = \int (\sinh - \cosh)^{2n} dt$$

$$= \int \frac{e^t - e^{-t} - e^t + e^{-t}}{2} dt = \int (e^{-t})^{2n} dt = -e^{-t} =$$

$$= -\frac{1}{2n} e^{-2nt} = -\frac{1}{2n} e^{-2n \operatorname{arcsinh} x} =$$

$$= -\frac{1}{2n} (x - \sqrt{1+x^2})^{-2n}$$

$$y_2 = -\frac{1}{2n} (x - \sqrt{1+x^2})^{2n} y_1 = -\frac{1}{2n} (x - \sqrt{1+x^2})^{2n} (x + \sqrt{1+x^2})^n =$$

$$= -\frac{1}{2n} (x - 1)^n (x - \sqrt{1+x^2}) = \frac{(-1)^{n+1}}{2} (x - \sqrt{1+x^2})^n$$

$$y_2 = (x - \sqrt{1+x^2})^n$$

$$y'' + py' + qy = 0 \quad \text{eva resten je } y_1$$

$$\text{Dalo si splatno formulo za } y_2 \quad w' + pw = 0$$

$$y_1 y_2' - y_1' y_2 = w$$

$$C'(x) y_1^2 = w$$

$$C(x) = \int \frac{w}{y_1^2} \quad y_2 = \int \frac{w}{y_1^2} dx \cdot y_1$$

b)

$$zy = y'$$

$$y'' = z'y + y'z$$

$$\begin{array}{c} z'y + y'z + pz y + qy = 0 \\ \parallel \\ yz^2 \end{array}$$

$$z'y + yz^2 + pz y + qy = 0$$

$$z' + z^2 + z = -q \quad \text{Riccatijeva enačba}$$

c) nestavek:  $z = u + \frac{1}{v} \quad u = \frac{y_1'}{y_1}$

$$z' = u' - \frac{1}{v^2} v'$$

$$u' - \frac{1}{v^2} v' + u^2 + 2\frac{u}{v} + \frac{1}{v^2} (u + \frac{1}{v}) - q = 0$$

$$\underbrace{u' + pu + q}_C + \left(-\frac{v'}{v^2}\right) + 2\frac{u}{v} + \frac{1}{v^2} + \frac{1}{v} = 0$$

$$-v' + 2uv + v + 1 = 0$$

$$-v^2 + v(-2u - p) - 1 = 0$$

$$\frac{v'}{v} = 2u + p \quad \ln v = \int 2u + p dt \quad v = e^{\int 2u + p dt}$$

$$w' + wp = 0 \Rightarrow w = e^{-\int p dt}$$

$$v = \frac{e^{\int 2u dt}}{w}$$

$$e^{\int 2u dt} = e^{2 \int \frac{y_1'}{y_1} dt} = y_1^2 \quad v = \frac{y_1^2}{w} = \frac{1}{C'(x)}$$

$$D'(x) \frac{y_1^2}{w} = 1 \Rightarrow D = \int \frac{w}{y_1^2} = C(x)$$

$$V(x) = \int \frac{w}{y_1} dt \frac{y_1^2}{w} + C \frac{y_1^2}{w}$$

$$\begin{aligned} z' &= \frac{y_1'}{y_1} + \frac{w}{y_1^2} \frac{1}{\int \frac{w}{y_1} dt} = \frac{y_1'}{y_1} + \frac{w}{y_1 y_2} = \frac{y_1' y_2 + y_1 y_2' - y_1' y_2}{y_1 y_2} \\ &= \frac{y_2'}{y_2} \end{aligned}$$

DN

Splana ravnina  $z' + az^2 + bz + c = 0$

$ayz = y'$  .... LDE ~~prej~~ red  
drugega

$$y'' + x^3 y = \sin x$$

$$y(0) = 1 \quad y'(0) = 0$$

za x na okolic 0

$$y(x) = \sum C_n x^n \quad \begin{matrix} C_0 = 1 \\ C_1 = 0 \end{matrix}$$

$$\sum_{k=2}^{\infty} k(k-1) C_k x^{k-2} + \sum_{k=0}^{\infty} C_k x^{k+3} = \sum_{x=0}^{\infty} (-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!}$$

$$2C_2 + 6C_3 x + 12C_4 x^2 + \sum_{k=3}^{\infty} (k+2)(k+1)(C_{k+2}, C_{k-3}) x^k$$

$$= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!}$$

~~$$V = \frac{1}{2} \int_{-\infty}^{\infty} \dots$$~~

$$C_2 = 0$$

$$C_3 = \frac{1}{6}$$

$$C_4 = 0$$

h<sub>1</sub>, h<sub>2</sub>:

$$(k+2)(k+1)(C_{k+2} + C_{k-3}) = \pm 1 \frac{1}{k!}$$

$$k = 2l+1$$

$$(2l+3)(2l+2)(C_{2l+3} + C_{2l-3}) = (-1)^{l+1} \frac{1}{(2l+1)!}$$

$$(2l+2)(2l+1)C_{2l+3} + 2C_{2l-3} = 0$$

$$C_{2l+3} = - \frac{C_{2l-3}}{(2l+3)(2l+2)}$$

$$C_{2l+3} = \frac{-C_{2l-3}}{(2l+2)(2l+1)}$$

# Variacijski račun

$$L: [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \quad L \in \mathcal{C}^2$$

$$\mathcal{A} = \{ y \in \mathcal{C}^2([a, b]) : y(a) = A, y(b) = B \}$$

$$\mathcal{L}: \mathcal{A} \rightarrow \mathbb{R}$$

$$\mathcal{L}(y) = \int_a^b L(x, y(x), y'(x)) dx$$

Iščemo kritične točke  $\mathcal{L}$

↳ Če je  $y_0 \in \mathcal{A}$  min/max  $\mathcal{L}$ , potem mora

$$\text{veljati } \left. \frac{d}{dt} \right|_{t=0} \mathcal{L}(y_0 + th) = 0 \quad (\text{za poljubno dopustno variacijo } h)$$

$\leadsto t \mapsto \mathcal{L}(y_0 + th)$  je običajna funkcija ene realne sprem.  $t$

Dopustna variacija  $h: y_0 + th \in \mathcal{A}$  vsaj za  $t \in (-\varepsilon, \varepsilon)$

$$\leadsto h \in \mathcal{C}^2([a, b])$$

$$h(a) = h(b) = 0, \text{ da bo } (y_0 + th)(a) = A$$

$$(y_0 + th)(b) = B$$

$$\forall t \in (-\varepsilon, \varepsilon)$$

$h \in \mathcal{H}$   
prostor dopustnih variacij

Naj bo  $L = L(x, t, s)$  Na dnu je pomembna formula

$\overset{''}{y}(t) \overset{''}{y}'(x)$

Računamo  $\frac{\partial}{\partial t} \Big|_{t=0} \int_a^b L(x, y+th, y'+th') dx = 0$

$$= \int_a^b \frac{\partial}{\partial t} \Big|_{t=0} L(\dots) dx = \int_a^b \left( h \underset{\frac{\partial L}{\partial t}}{L_t}(x, y_0, y_0') + h' \underset{\frac{\partial L}{\partial s}}{L_s}(x, y_0, y_0') \right) dx$$

$$= \int_a^b (L_y \cdot h + L_{y'} \cdot h') dx = \int_a^b L_y h dx + \int_a^b L_{y'} h' dx =$$

$$u = L_y \quad v = h$$

$$du = \frac{\partial}{\partial x} L_y dx \quad dv = h' dx$$

$$\frac{\partial}{\partial x} L_y'(x, y_0, y_0')$$

$$\int L_{y'} h' dx = L_{y'} h \Big|_a^b - \int_a^b h L_{y'}' dx$$

$$DL_{y_0}(h) = L_{y'} h \Big|_a^b + \int_a^b \left( L_y - \frac{\partial}{\partial x} L_{y'} \right) h dx = 0$$

$$L_{y'}(b, y_0'(b), y_0'(b)) h(b) - \dots - \underbrace{h(a)}_{=0}$$

$$\int_a^b \left( L_y - \frac{\partial}{\partial x} L_{y'} \right) h dx = 0 \quad \text{za } \forall h \in \mathcal{H}$$

osnovna lema  
o variacijskem računu

$$\Rightarrow L_y - \frac{\partial}{\partial x} L_{y'} \equiv 0$$

to je Euler-Lagrangeov pogoj

Uredimo oznake:

$$EL = L_y - \frac{\partial}{\partial x} L_{y'}$$

$$L_y(x, y(x), y'(x)) - \frac{\partial}{\partial x} L_{y'}(x, y(x), y'(x)) = 0$$

$\hookrightarrow$  v splošnem je to DE II. reda

① Določiti kritične točke

a)  $\mathcal{L}(y) = \int_0^1 (y'^2 + y y' + 12xy) dx \quad y(0) = y(1) = 0$

b)  $\mathcal{L}(y) = \int_0^1 (e^y + x y') dx \quad y(0) = 0 \quad y(1) = a \in \mathbb{R}$

c)  $\int_{-1}^1 \sqrt{1+x^2+y'^2} dx \quad y(-1) = 0 \quad y(1) = 1$

a)  $L = y'^2 + y y' + 12xy$

$L_y = \cancel{y''} + y' + 12x$

$L_{y'} = 2y' + y$

$L_y - \frac{\partial}{\partial x} L_{y'} = 0$  ← tu upoštevamo da je  $y(x)$

$y' + 12x - 2y'' - y' = 0$

$y'' = 6x$

$y' = 3x^2 + D$

$y = x^3 + Dx + C$

$y(0) = C = 0$

$y(1) = 1 + D = 0 \Rightarrow D = -1$

$y = x^3 - x$

Delamo se da so

$y, y'$  x neodvisne,  
samo oznake

$$b) \mathcal{L}(y) = \int_0^1 (e^y + xy') dx \quad y(0)=0 \quad y(1)=a \in \mathbb{R}$$

$$L = e^y + xy'$$

$$L_y = e^y$$

$$L_{y'} = x$$

$$L_y - \frac{\partial}{\partial x} L_{y'} = 0$$

$$e^y - 1 = 0$$

$$y = \ln 1 = 0$$

$$y \equiv 0$$

$$y(1) = a = 0$$

$a=0 \Rightarrow$  potem ima  $\mathcal{L}$  kritično točko,  
drugače pa nima lokalnih ekstremov

$$c) \int_{-1}^1 \sqrt{1+x^2+y'^2} dx \quad y(-1)=0 \quad y(1)=1$$

$$L = \sqrt{1+x^2+y'^2}$$

$$L_y = 0$$

$$L_{y'} = \frac{y'}{\sqrt{1+x^2+y'^2}}$$

$$y' \cdot y'$$

//

$$L_y - \frac{d}{dx} L_{y'} = - \frac{y''(\sqrt{\dots}) - (x+y'^2)}{(1+x^2+y'^2)^{3/2}} = 0$$

$$= \frac{y''(1+x^2+y'^2) - x + y'^2}{(1+x^2+y'^2)^{3/2}} = 0$$

$$\Leftrightarrow y''(1+x^2+y'^2) - x + y'^2 = 0$$

$$-\frac{d}{dx} L_{y'} = 0 \Rightarrow L_{y'} = C$$

$$\frac{y'}{\sqrt{1+x^2+y'^2}} = C$$

$$y'^2 = C(1+x^2+y'^2)$$

$$y'^2(1-C) = (1+x^2)C$$

$$y' = \sqrt{\frac{(1+x^2)C}{1-C}}$$

$$y = \int \underbrace{\sqrt{\frac{C}{1-C}}}_{D'} \underbrace{\sqrt{1+x^2}}_{\text{ch} x \text{ nach}} dx = \text{sh} x \sqrt{\frac{C}{1-C}} + D$$

$$x = \text{sh} t$$

$$dx = \text{ch} t$$

$$y = D \int \text{ch}^2 t dt = \frac{D}{2} \left( \frac{1}{2} \text{sh} 2t + t \right) + E =$$

$$\text{ch}^2 t = \frac{\text{ch} 2t + 1}{2}$$

$$= \frac{D}{2} (x\sqrt{1+x^2} + \ln|x+\sqrt{1+x^2}|) + E$$

$$y(-1) = \frac{D}{2} (-\sqrt{2} + \ln|-1+\sqrt{2}|) + E = 0$$

$$y(1) = \frac{D}{2} (\sqrt{2} + \ln|1+\sqrt{2}|) + E = 1$$

$$\frac{D}{2} \ln(2-1) + 2E = 1 \Rightarrow E = \frac{1}{2}$$

$$D(-\sqrt{2} + \ln|\sqrt{2}-1|) = -1$$

$$D = \frac{-1}{-\sqrt{2} + \ln|\sqrt{2}-1|}$$

$$\frac{D}{2} (2\sqrt{2} + \ln|\frac{1+\sqrt{2}}{1-\sqrt{2}}|) = 1$$

$$\frac{D}{2} (2\sqrt{2} + \ln \frac{(1+\sqrt{2})^2}{1}) = 1$$

$$D = \frac{1}{\sqrt{2} + 2 \ln(1+\sqrt{2})}$$

Maj bo  $L = L(y, y')$  neodvisen od  $x$

A) Pokaži da je  $y \in \tilde{C}$  kritična teče funkcionala

$$\mathcal{L}(y) = \int_a^b L(y, y') dx \quad y(a) = A \quad y(b) = B$$

$$\Leftrightarrow L - y' L_{y'} = C \in \mathbb{R} \quad \text{Beltramijeva identiteta}$$

b) Dokaži kritična teče  $\int_a^b y \sqrt{1-y'^2} dx \quad y(a) = y(b) = A$

$$(\Leftrightarrow) L_y - \frac{\partial}{\partial x} L_{y'} = 0$$

$$L - y' L_{y'} = C \quad / \quad \frac{\partial}{\partial x} \quad L(x, y(x), y'(x)) = L(y(x), y'(x))$$

$$L_y \cdot y' + L_{y'} y'' - y'' L_{y'} - y' \frac{\partial}{\partial x} L_{y'} = 0$$

$$\underset{0}{y'} (L_y - \frac{\partial}{\partial x} L_{y'}) = 0 \Leftrightarrow L_y - \frac{\partial}{\partial x} L_{y'} = 0$$

$$b) y \sqrt{1-y'^2} - y' y \frac{-y'}{\sqrt{1-y'^2}} = C$$

$$y(1-y'^2) + y'^2 y = C \sqrt{1-y'^2}$$

$$y = C \sqrt{1-y'^2}$$

$$y^2 = (1-y'^2) C$$

$$y'^2 = 1 - \frac{y^2}{C}$$

$$y' = \sqrt{1 - \frac{y^2}{C}} \quad \dots \quad y' = \sqrt{c^2 y^2 - 1}$$

$$y = \int \frac{dx}{\sqrt{c^2 y^2 - 1}} = \int dx = x + D$$

$$c y = \sinh t$$

$$\sinh t dt = c y' dx$$

$$= \frac{1}{c} \int dt = \frac{1}{c} t = \frac{1}{c} \operatorname{arch}(cy) = x + D$$

$$t = cx$$

$$y = \frac{1}{c} \operatorname{ch}(cx + D)$$

Dolci i drugi smerni odvod funkcionala

$$I(y) = \int_a^b L(x, y, y') dx$$

$$QI_y(h) = \frac{\partial^2}{\partial t^2} \bigg|_{t=0} I(y+th) = \frac{\partial^2}{\partial t^2} \bigg|_{t=0} \int_a^b L(x, y+th, y'+th') dx =$$

$$= \frac{d}{dt} \bigg|_{t=0} \int_a^b h \cdot L_y(x, y+th, y'+th') + h' L_{y'}(x, y+th, y'+th') dx =$$

$$= \int_a^b h(h L_{yy}(x, y+th, y'+th') + h' \dots \dots \dots) \bigg|_{t=0} =$$

$$= \int_a^b h^2 L_{yy}(x, y, y') + 2hh' L_{yy'}(\dots) + h'^2 L_{y'y'}(\dots) dx$$

$\parallel$   
 $(h^2)'$

perpetres  $\rightarrow$

$$\int_a^b h^2 (L_{yy} - \frac{d}{dx} L_{yy'}) + h'^2 L_{y'y'} dx$$

$y_0$  kritična in  $QI_{y_0}(h) > 0$  (minimum) ali

$QI_{y_0}(h) < 0$  (maksimum)

$\Rightarrow y_0$  je minimum/maksimum

✓ nekaj  
nalog z  
vsebinski odvod:

$$\mathcal{A} = \{x \mapsto (\overbrace{y(x), z(x)}^y); \vec{y}(a) = \vec{A}, \vec{y}(b) = \vec{B}\} \subset C^2([a, b], \mathbb{R}^2)$$

$$\mathcal{L}(y, z) = \int_a^b L(y, y', z, z') dx$$

Dopustne varijacije  $(h, k)$

$$\begin{aligned} h(a) &= 0 \\ h(b) &= 0 \\ k(a) &= k(b) = 0 \end{aligned}$$

EL požj je v tem kontekstu

$$\bullet L_y - \frac{d}{dx} L_{y'} = 0$$

$$\bullet L_z - \frac{d}{dx} L_{z'} = 0$$

3015 ei k. d. one take

$$L(y, z) = \int_0^{\frac{\pi}{2}} (y'^2 + z'^2 - 2yz) dx = 0$$

$$\begin{aligned} y(0) &= 0 \\ y(\frac{\pi}{2}) &= 1 \\ z(0) &= 0 \\ z(\frac{\pi}{2}) &= 1 \end{aligned}$$

$$L_y = -2z$$

$$L_{y'} = 2y' \quad -2z - \frac{\partial}{\partial x} 2y' = 0$$

$$L_{z'} = -2y \quad -2y - \frac{\partial}{\partial x} 2z' = 0$$

$$L_z = 2z'$$

$$-2z - 2y'' = 0$$

$$-2y - 2z'' = 0$$

$$z + y'' = 0$$

$$y + z'' = 0$$

$$z = -y''$$

$$z' = -y'''$$

$$y'''' = y$$

$$\lambda^4 - 1 = 0 \quad (\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i)$$

$$\Rightarrow y = Ae^x + Be^{-x} + C\cos x + D\sin x$$

$$y' = Ae^x - Be^{-x} - C\sin x + D\cos x$$

$$z = -y'' = -Ae^x - Be^{-x} + C\cos x + D\sin x$$

$$y(0) = A + B + C = 0$$

$$y(\frac{\pi}{2}) = Ae^{\frac{\pi}{2}} + Be^{-\frac{\pi}{2}} + D = 1$$

$$z(0) = -A - B + C = 0$$

$$z(\frac{\pi}{2}) = -Ae^{\frac{\pi}{2}} - Be^{-\frac{\pi}{2}} + D = 1 \quad \Rightarrow C = 0$$

$$A + B = 0 \Rightarrow A = -B$$

$$2D = 1 \Rightarrow D = \frac{1}{2}$$

$$A + B = 0$$

$$A(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}) = \frac{1}{2}$$

$$A = \frac{1}{2} \frac{1}{e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}}$$

??

o3 dano:

$$A = B = C = 0 \quad D = 1$$

$$z(x) = \sin x$$

$$y(x) = \sin x$$