

Definicije

Dobro postavljen problem (PDE + pogoji)

1. Rešitev obstaja
2. Rešitev je ena sama
3. Rešitev je zvezno odvisna od začetnih pogojev (stabilnost)

Karakteristično polje (a, b, c)

Pogoj transversalnosti $\begin{vmatrix} f'(s) & a(p) \\ g'(s) & b(p) \end{vmatrix} \neq 0$

Linearni diferencialni operatorji: $\delta = b^2 - ac$

• eliptična $\delta < 0 \leadsto W_{zz} + W_{\eta\eta}$

• parabolčna $\delta = 0 \leadsto W_{zz}$

• hiperbolična $\delta > 0 \leadsto W_{z\eta}$

D'Alembertova formula

$$u(x, t) = \langle f \rangle_{\partial I} + t \langle g \rangle_I + \frac{t^2}{2} \langle \varphi \rangle = \frac{1}{2} (f(x+ct) + f(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z) dz + \frac{1}{2} \int_{\Delta} \varphi ds$$

Poissonovo jedro $P_r(s) = \frac{1-r^2}{1-2r\cos(\pi s)+r^2} = \frac{1-|z|^2}{|1-z|^2}$

Drugo jedro: $Q_r(s) = \frac{2|\ln|z||}{|1-z|^2} = \frac{2r\sin(\pi s)}{1-2r\cos(\pi s)+r^2}$

$$Q_1 = \begin{cases} 0 & s=0 \\ \cot(\pi s) & s \neq 0 \end{cases}$$

Hilbertova transformiranka $H_f(t) = \lim_{\delta \rightarrow 0} \int_{\delta < |s|} f(t-s) \cot(\pi s) ds$

Neko splošno jedro $P_z(z, w) = \frac{1}{2\pi} P_r(z \overline{\Psi(w)}) \Psi(z)$

Greenova reprezentacijska formula

$$u(z) = \int_{\partial M} (u \cdot \nu_K - \nu u \cdot K)(z, \cdot) dS + \iint_M K \cdot \Delta u(z, \cdot) dS$$
$$K = \frac{1}{2\pi} \ln |z - z|$$

Greenova funkcija $G = K + h$, kjer je h

take harmonična funkcija, daje $K + h = 0$ na ∂M

$$h(z, z) = \int_{\partial M} K(w, z) P_M(z, w) dw$$

$$\rho_{\alpha, \beta} = \sup |x^\alpha \cdot \partial^\beta f| = \sup \left| \prod_{i=1}^n x_i^{\alpha_i} \cdot \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}} \right|$$

Schwartzov razred $\mathcal{S}(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) : \forall \alpha, \beta, \rho_{\alpha, \beta}(f) < \infty \}$

Fourierjeva transformacija $\hat{f}(z) = \int_{\mathbb{R}^n} f(z) e^{-2\pi i \langle z, z \rangle} dz$

Plancherelova identiteta $\|f\|_2 = \|\hat{f}\|_2 = \|\check{f}\|_2$

Začetne trditve

Str:

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Trditev:

- $U \subseteq \mathbb{R}^n$ odprta okolica ničle
- $f \in C^2(U)$

Potem je
$$\Delta f(0) = \lim_{r \searrow 0} \frac{2n}{r^2} \frac{1}{\omega(S_r)} \int_{S_r} (f(x) - f(0)) d\omega(x)$$
$$= \langle f \rangle_{S_r} - f(0)$$

Dokaz:

1. Taylorjeva vrsta za $f = \mathbf{1} + \frac{1}{2} \mathbf{1} + o(|x|^2)$

2. integriramo

- $\int_{S_r} x_j d\omega = 0$ (lihost)
- $\int_{S_r} x_j x_k d\omega = 0$ (lihost za $j \neq k$)
- $\int_{S_r} x_k^2 d\omega = \frac{1}{n} \int_{S_r} x_1^2 + \dots + x_n^2 d\omega$ (simetričnost)

3.
$$\Delta f(0) = \sum \frac{\partial^2 f(0)}{\partial x_i^2}$$

□

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Trditev: Laplacev operator je invarianten za toge premike in rotacije.

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KVAZILINEARNE ENAČBE 1. reda

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

Ploskev: $(a, b, c) \cdot (u_x, u_y, -1) = 0$

$$\vec{r}(x,y) = (x, y, u(x,y))$$

$$\text{normala} = \vec{r}_x \times \vec{r}_y = -(u_x, u_y, -1)$$

$(a, b, c)(p)$ podaja tangentno ravnino

$$\gamma(t) = (x, y, z) \quad (\gamma(t_0) \in (a, b, c)(p_0), \gamma \subseteq \{z = u(x,y)\})$$

$$\Rightarrow \dot{x} = a \quad \dot{y} = b \quad \dot{z} = c \quad - \text{smer odvoda krivulje}$$

Eksistenčni izrek

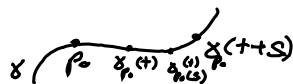
• V enakomerno Lipschitzova, zveza

$$\dot{z} = V(z) \quad z(0) = z_0$$

Potem obstaja enolična rešitev v neki okolici začetnega pogoja.

$$\dot{\gamma} = V(\gamma) \quad (V = (a, b, c)) \quad \gamma(0) = p_0 = \gamma_{p_0}$$

$$\text{Izkaže se: } \gamma_{p_0}(t+s) = \gamma_{\gamma_{p_0}(s)}(t)$$



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Trditva

$$\bullet a, b, c \in C^1(\mathbb{R}^3) \quad \bullet S = \{z = u(x, y)\}$$

$$\bullet p_0 = (x_0, y_0, z_0) \in S \quad (z_0 = u(x_0, y_0))$$

Potem vsake karakteristične krivulja za KLE
cela leži v S .

Dokaz:

$$\text{Dokazujemo } \forall t. \quad z(t) = u(x(t), y(t))$$

$$\text{oziramo } \underline{w(t) = z(t) - u(x(t), y(t)) \equiv 0}$$

$$\dot{w}(t) = \dot{z} - \dot{x}u_x - \dot{y}u_y =$$

$$= -a(x, y, z)u_x - b(x, y, z)u_y + c(x, y, z) =$$

$$= -a(x, y, w + u(x, y))u_x - b(x, y, w + u)u_y + \dots$$

$$= f(t, w)$$

$$w(0) = 0$$

Ali: lahko uporabimo eksistenčni izrek?

Lipschitzovost: zvezno odvedljiva, omejena

$\Rightarrow$ je Lipschitzova

Uporabimo eksistenčni izrek:

$w \equiv 0$ reši enačbo. zaradi enoličnosti

je potem $z(t) = u(x(t), y(t))$

□

Posledica trditve: integralna ploskev
je unija karakterističnih funkcij
(skozi vsako točko na ploskvi lahko potegnemo
karakteristiko)

Posledica:

- $a, b, c \in C^1(\mathbb{R}^3)$
- $S = \{z = u(x, y)\}$
- $p_0 = (x_0, y_0, z_0) \in S$ ($z_0 = u(x_0, y_0)$)

Potem:

1. $p \in S_1 \cap S_2 \Rightarrow$ cela tokovnica skozi p leži v $S_1 \cap S_2$
2. $S_1 \cap S_2$ je krivulja $\Rightarrow S_1$ in S_2 se NE
dotikata in $S_1 \cap S_2$ je integralna krivulja

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Cauchyjev problem za KLE

Iščemo integralno ploskev, ki vsebuje krivuljo, ki je začetni pogoj.

$$R(s, t) = (x(s, t), y(s, t), z(s, t)) = ?$$

$$R(s, 0) = (f(s), g(s), h(s)) \leftarrow$$

Fiksiramo s in režimo problem

$$au_x + bu_y = c \leadsto \dot{R}_s = V(R_s) \text{ in } R_s(0) = R(s, 0)$$

Ekstistenčni izrek $\leadsto$ Imamo $R_s(t)$

Kdaj lahko to sestavimo v ploskev $R(s, t)$?

Poglejmo kdaj $\{R(s, t)\}$ določa graf $\{z = u(x, y)\}$

$x(s, t), y(s, t) \leadsto$ izrek o inverzni funkciji.

Rabimo pogoj $\frac{\partial(x, y)}{\partial(s, t)}(s_0, 0) \neq 0$

$$\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} (s_0, 0) = \begin{vmatrix} f'(s_0) & a(p_0) \\ g'(s_0) & b(p_0) \end{vmatrix} \neq 0$$

(pogoj transverzalnosti)

Torej v okolici x_0, y_0 obstaja rešitev.

Pokažimo še enoličnost:

Rečimo da u in w rešita problem

S_1 za u in S_2 za w se sekata v začetnih

pogojih. Torej so vse tokornice vsebovane

$$v S_1 \text{ in } v S_2 \Rightarrow S_1 = S_2$$

NELINEARNE ENAČBE 1. REDA

$$F(x, y, u, u_x, u_y) = 0 \leadsto F(x, y, u, p, q) = 0$$

Recimo da u , ki reši problem že obstaja.

$S = \{z = u(x, y)\}$ je ploskev z normalo

$$\vec{n} = (u_x, u_y, -1) = (p, q, -1)$$

$\vec{n}(\lambda) = (p(\lambda), q(\lambda), -1)$ družine norm, da

če p in q velja $F(x, y, u, p(\lambda), q(\lambda)) = 0$

že Mongojev stožec ustreza enačbama

$$\langle \vec{r} - \vec{r}_0, \vec{n}(\lambda) \rangle = 0 \quad \langle \vec{r} - \vec{r}_0, \vec{n}'(\lambda) \rangle = 0$$

$$\vec{n}(\lambda) \times \vec{n}'(\lambda) = (q'(\lambda), -p'(\lambda), p q'(\lambda) - p' q(\lambda))$$

Odvajamo F po λ

$$\frac{\partial}{\partial \lambda} F = p' F_p(x, y, z, p, q) + q' F_q(x, y, z, p, q) = 0$$

$$(p', q') \perp (F_p, F_q) \Rightarrow (F_p, F_q) \parallel (q', -p')$$

$$\Rightarrow (F_p, F_q, \underbrace{p F_p + q F_q}_{\text{ohranjanje razmerij } (a, b, pa+qb)}) \parallel (q', -p', \underbrace{p q' - p' q}_{\text{ohranjanje razmerij } (a, b, pa+qb)}) = \vec{r} - \vec{r}_0$$

Parametrizacija stožca

$$r(\lambda) = \vec{r}_0 + \mu \cdot (F_p(\lambda), F_q(\lambda), p F_p + q F_q)$$

$\gamma(t) = (x, y, z) \in S \Rightarrow \ddot{\gamma}(t) \in \text{Mongojev stožec}$

$$\Rightarrow \text{zahtevamo } \dot{x} = F_p, \dot{y} = F_q, \dot{z} = p F_p + q F_q$$

Trditveni:

$$\bullet \gamma(t) = (x, y, z) \quad \bullet \dot{z} = p\dot{x} + q\dot{y}$$

Potem je tangenta na družino krivulj in družino ravnin skozi njene točke določena s p in q

$$\text{Dokaz: } \langle \dot{\gamma}, (p, q, -1) \rangle = \langle (\dot{x}, \dot{y}, p\dot{x} + q\dot{y}), (p, q, -1) \rangle = 0$$

□

$F=0$ odvajamo po x in y in p in q po t

$$F_x + u_x F_u + p_x F_p + q_x F_q = 0$$

$$p_x F_p = -(F_x + u_x F_u + q_x F_q)$$

$$F_y + u_y F_u + p_y F_p + q_y F_q = 0$$

$$q_y F_q = -(F_y + u_y F_u + p_y F_p)$$

$$\begin{aligned} \dot{p} &= \dot{x} p_x + \dot{y} p_y = p_x F_p + p_y F_q = \\ &= F_q (p_y - q_x) - (F_x + u_x F_u) \end{aligned}$$

$$\text{Želimo } (u_x)_y = (u_y)_x \leadsto p_y = q_x$$

$$\text{Torej } \dot{p} = -(F_x + p F_u)$$

$$\dot{q} = -(F_y + q F_u)$$

Torej vse skupaj imamo

$$\dot{x} = F_p$$

$$\dot{y} = F_q$$

$$\dot{z} = p F_p + q F_q$$

$$\dot{p} = -(F_x + p F_u)$$

$$\dot{q} = -(F_y + q F_u)$$

Trditev:

F je konstantna vzdolž vsake rešitve sistema

$$\dot{x} = F_p$$

$$\dot{p} = -(F_x + p F_z)$$

$$\dot{y} = F_g$$

$$\dot{g} = -(F_y + g F_z)$$

$$\dot{z} = p F_p + g F_g$$

Dokaz: $\frac{\partial}{\partial t} F(x, y, z, p, g) = \dots = 0$

□

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Cauchyjeva naloga za PDE

Integralne ploskev za $F(x, y, z, p, g) = 0$ vsebuje krivuljo $\gamma(s) = (f(s), g(s), h(s))$

5 enačb, 3 pogoji $\Rightarrow$ iščemo še ϕ in ψ pogoja

$$R(s, t) = (x, y, z, p, g)$$

$$R(s, 0) = (f, g, h, \phi, \psi)$$

$$\text{Želimo } z = u(x, y), u_x = p, u_y = g \Rightarrow \dot{z} = \dot{x}p + \dot{y}g$$

$$\leftarrow z_s = x_s p + y_s g$$

$$\text{Vstavimo } t=0: h' = f' \phi + g' \psi$$

$$\Rightarrow \begin{bmatrix} z_s \\ z_t \end{bmatrix} = \underbrace{\begin{bmatrix} x_s & y_s \\ x_t & y_t \end{bmatrix}}_A \begin{bmatrix} p \\ g \end{bmatrix}$$

$$\det A \neq 0 \Rightarrow \begin{bmatrix} p \\ g \end{bmatrix} = A^{-1} \begin{bmatrix} z_s \\ z_t \end{bmatrix} = \begin{bmatrix} s_x & t_x \\ s_y & t_y \end{bmatrix} \begin{bmatrix} z_s \\ z_t \end{bmatrix} = \begin{bmatrix} z_x \\ z_y \end{bmatrix}$$

$\gg$

Rešujemo torej sistem:

$$\dot{x} = F_p \quad \dot{p} = -(F_x + pF_z)$$

$$\dot{y} = F_g \quad \dot{g} = -(F_y + gF_z)$$

$$\dot{z} = pF_p + gF_g \quad \begin{vmatrix} p' & g' \\ F_p & F_g \end{vmatrix} \neq 0$$

$$h' = \phi p' \cdot \psi g'$$

Eksistenčni izrek $\leadsto \exists R(s, t). R(s, 0) = (p, g, h, \phi, \psi)$

in (x, y, z) so lokalno graf $u(x, y)$

Ali: x, y, z, p, g rešijo $F(x, y, u, u_x, u_y) = 0$

Preverimo $p = u_x$ in $g = u_y$

$$\begin{bmatrix} z_s \\ z_t \end{bmatrix} = \begin{bmatrix} x_s & y_s \\ x_t & y_t \end{bmatrix} \begin{bmatrix} z_x \\ z_y \end{bmatrix} = A \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

če velja $\begin{bmatrix} z_s \\ z_t \end{bmatrix} = A \begin{bmatrix} p \\ g \end{bmatrix}$ in A obrnljiva

$$\Rightarrow p = u_x, g = u_y$$

Torej dokazujemo $z_s = p x_s + g y_s$

$$37 \quad B(s, t) = z_s - p x_s - g y_s$$

$$\begin{aligned} \frac{\partial B}{\partial t} &= \dots = \frac{\partial}{\partial s} (z_t - p x_t - g y_t) + p_s x_t + g_s y_t - p_t x_s - g_t y_s = \\ &= F_p p_s + F_g g_s + (F_x + p F_z) x_s + (F_y + g F_z) y_s = \\ &= \dots = -F_z \cdot B \end{aligned}$$

$$\leadsto B(s, t) = B(s, 0) e^{\int_0^t \dots dt} = 0$$

LINEARNI DIFERENCIALNI OPERATORJI

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$$

$$M = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad \delta = -\det$$

eliptična $\delta < 0$

hiperbolična $\delta > 0$

parabolična $\delta = 0$

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} = \begin{bmatrix} z_x & z_y \\ \eta_x & \eta_y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} z_x & \eta_x \\ z_y & \eta_y \end{bmatrix}$$

Izrek

• $L(u) = au_{xx} + 2bu_{xy} + cu_{yy} + \dots = g$ hiperbolična

Potem $\exists z, \eta$ koordinate. $w_{z\eta} + h_1(w) = \tilde{g}$

Dokaz: (ideja)

• želimo $A = C = 0$

• Zapišemo v kvadratni enačbi za $\frac{z_x}{z_y}$

• $a z_x + (b + \sqrt{b^2 - ac}) z_y = 0 \Rightarrow$ linearna prva red za $z(x, y)$

$$\dot{x} = a \quad \dot{y} = b + \sqrt{b^2 - ac} \quad \dot{z} = 0$$

$$\Rightarrow \frac{\partial y}{\partial x} = \frac{\dot{y}}{\dot{x}} \Rightarrow y = y(x; C) \Rightarrow C = C(x, y) = z(x, y) \quad \square$$

Izrek (Vmesni)

- $a(x,y), b(x,y)$ zvezno parcialno odvedljivi
- $F = \frac{b}{a}$
- Φ prvi integral za $y' = F(x,y)$

Potem Φ reš: $au_x + bu_y = 0$

Dokaz:

$$\begin{aligned}\Phi(x, y(x)) &\equiv \text{const} \quad \leftarrow \text{obstaja po desistencnem izreku za Cauchyjev problem} \\ 0 &= \frac{\partial}{\partial x} \Phi(x, y(x)) = \Phi_x + y' \cdot \Phi_y = \Phi_x + F \Phi_y \\ &\Rightarrow a \Phi_x + b \Phi_y = 0\end{aligned}$$

□

Izrek

- $L(u) = au_{xx} + bu_{xy} + cu_{yy} + \dots = g$ parabolichna

Potem $\exists z, \eta$ koordinate. $LW = A w_{zz}$

Dokaz: ($B=C=0 \Rightarrow$ dovolj $C=0$)

$$\frac{b^2}{a} = c \Rightarrow C = \frac{1}{a} (az_x + bz_y)^2$$

Rešujemo $az_x + bz_y = 0$ linearne 1. reda

$$\dot{x}=a \quad \dot{y}=b \quad \dot{z}=0 \Rightarrow \frac{\partial y}{\partial x} = \frac{\dot{y}}{\dot{x}} = \frac{b}{a}$$

Vmesni izrek: Naj bo η prvi integral z $y' = \frac{b}{a}(x,y)$

$\Rightarrow a\eta_x + b\eta_y = 0$ za z vzamemo x

$$\begin{vmatrix} 1 & 0 \\ \eta_x & \eta_y \end{vmatrix} = \eta_y \quad \text{če } \eta_y = 0 \Rightarrow \eta_x = 0$$

$\uparrow$
 $a\eta_x + b\eta_y = 0$

ker je η netrivialen 1. integral $\Rightarrow \eta_y \neq 0$

□

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Izrek

• $L(u) = au_{xx} + bu_{xy} + cu_{yy} + \dots = g$ eliptična

Potem $\exists z, \eta$ koordinate. $LW = A(W_{zz} + W_{\eta\eta})$

Dokaz:

$$\text{Želimo } A=C \quad B=0 \Rightarrow A-C+2 \cdot B=0$$

$$\begin{aligned} A-C+2 \cdot B &= \dots = a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 \quad \phi = z + i\eta \\ &= (a\phi_x + (b+i\sqrt{a})\phi_y)(a\phi_x + (b-i\sqrt{a})\phi_y) \end{aligned}$$

$$\Rightarrow a\phi_x + (b+i\sqrt{a})\phi_y = 0$$

$$\dot{x} = a \quad \dot{y} = b + i\sqrt{a} \quad \frac{\partial x}{\partial y} = \frac{\dot{x}}{\dot{y}} = \frac{b+i\sqrt{a}}{a} =: F$$

$F: \mathbb{R} \rightarrow \mathbb{C}$ želimo $F: \mathbb{C} \rightarrow \mathbb{C}$ zakej? ker bomo

želeli uporabiti dejstvo, da je Φ holomorfn.

F razširimo na holomorfn z Taylorjavo vrsto.

Φ naj bo prvi integral za funkcijo $y' = F(x, y(x))$ in Φ je holomorfn, ker je F holomorfn (se mi ed).

$$\Rightarrow z_x = \eta_y \quad z_y = -\eta_x$$

Preverimo, da so to res koordinate

$$\begin{vmatrix} z_x & z_y \\ \eta_x & \eta_y \end{vmatrix} = z_x \eta_y - z_y \eta_x = z_x^2 + z_y^2 = |\nabla z|^2 = |\nabla \eta|^2 \neq 0$$

□

56 VALOVNA ENAČBA NA REALNI OSI

$$u_{tt} - c^2 u_{xx} = 0$$

$$\Delta = c^2 > 0 \Rightarrow \text{hiperbolične} \Rightarrow w_z \eta = 0 \Rightarrow w = F(\xi) + G(\eta)$$

$$\xi = x - ct \quad \eta = x + ct$$

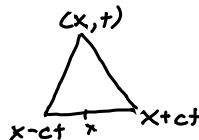
57 Cuchyjev problem $u(x, 0) = f(x) \quad u_t(x, 0) = g(x)$

$$F(x) + G(x) = f(x)$$

$$-F(x) + G(x) = \frac{1}{c} \int_0^x g(z) dz + d$$

$$\Rightarrow u(t, x) = \frac{1}{2} (f(x-ct) + f(x+ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z) dz$$

karakteristiki; trikotnik



Cauchyjev problem za nehomogeno valovno enačbo

$$u_{tt} - c^2 u_{xx} = f \quad u(x, 0) = f(x) \quad u_t(x, 0) = g(x)$$

Eksistenčni izrek za nehomogeno valovno enačbo

Rešitev 1. obstaja

2. je enolična

3. je zvezo odvisna od začetnih pogojev

Dokaz:

$$\begin{aligned} \int_{\partial \Delta} (Q_t - P_x) dV &= \int_{\partial \Delta} P dt + Q dy \\ 1. \quad u_{tt} - c^2 u_{xx} &= f \Rightarrow \int_{\Delta} (u_{tt} - c^2 u_{xx}) dS = \int_{\Delta} f dS \\ &\Rightarrow \int_{\Delta} f dS = \int_{\partial \Delta} (u_t dx + c^2 u_x dt) = \\ &= \int_{\partial \Delta} (u_t, c^2 u_x) d\vec{S} = \dots = \\ &= - \int_{x-ct}^{x+ct} g(z) dz + 2c u(x, t) - c(f(x+ct) + f(x-ct)) \end{aligned}$$

$$\begin{aligned} \Rightarrow u(x, t) &= \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z) dz + \frac{1}{2} \int_{\Delta} f dS \\ &= \langle f \rangle_{\partial I} + t \langle g \rangle_I + \frac{t^2}{2} \langle f \rangle_{\Delta} \end{aligned}$$

D'Alembertova formula

2. sledi iz 1.

$$3. \text{ Recimo da } \|f_1 - f_2\|_{\infty}, \|f_1 - f_2\|_{\infty}, \|g_1 - g_2\|_{\infty} < \delta$$

$$\begin{aligned} u_1 - u_2 &= \langle f_1 - f_2 \rangle_{\partial I} + \langle g_1 - g_2 \rangle_I + \frac{t^2}{2} \langle f_1 - f_2 \rangle_{\Delta} \\ &\leq \delta + t\delta + \frac{t^2}{2} \delta \leq \varepsilon \end{aligned}$$

$$\Rightarrow \delta \leq \varepsilon \frac{1}{1+t+\frac{t^2}{2}}$$

gledamo lokalno za $t \in [0, T]$

$$\Rightarrow \delta = \varepsilon \frac{1}{1+T+\frac{T^2}{2}}$$

□

60 TOPLOTNA ENAČBA (cauchyjev problem)

$$u_t - \Delta u = 0 \quad \text{na } U \times (0, \infty)$$

$$u = 0 \quad \text{na } \partial U \times [0, \infty)$$

$$u = g \otimes 1 \quad \text{na } U \times \{0\}$$

Separacija spremenljivk $\leadsto \frac{\Delta X}{X}(x) = \frac{T'}{T} =: -\lambda$

$\leadsto T(t) = C e^{-\lambda t}$, X je lastni vektor za $-\Delta$

$$u(x, 0) = X(x) = g(x) \quad \leadsto g \text{ ni nujno lastni vektor za } -\Delta$$

Gledamo za $n=1$

$$\leadsto \frac{X''}{X} = -\lambda \leadsto X = A \cos(\sqrt{\lambda} x) + B \sin(\sqrt{\lambda} x)$$

$$u = 0 \quad \text{na } \partial U \times [0, \infty) \Rightarrow X \equiv 0 \quad \text{na } \partial \mathcal{R} \quad U = [a, b]$$

$$\begin{bmatrix} \cos(\sqrt{\lambda} a) & \sin(\sqrt{\lambda} a) \\ \cos(\sqrt{\lambda} b) & \sin(\sqrt{\lambda} b) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

$$\text{predpogoj: } \det(\dots) = 0 \quad \det(\dots) = \sin(\sqrt{\lambda}(b-a))$$

$$\leadsto \sqrt{\lambda}(b-a) = k\pi \leadsto \lambda = \left(\frac{k\pi}{b-a}\right)^2 \quad k \in \mathbb{N}_0$$

Poglejmo še $\lambda > 0$

$$\begin{aligned} -\Delta X &= \lambda X \leadsto \langle -\Delta X, X \rangle = \int_U (-\Delta X) \cdot X \, dW = \\ &= \int_U \nabla X \cdot \nabla X \, dW + \int_{\partial \mathcal{R}} X \cdot \nabla X \, d\vec{s} = \int_U |\nabla X|^2 \geq 0 \\ &= \langle \lambda X, X \rangle = \lambda \|X\|^2 \geq 0 \quad X \equiv 0 \quad \text{na } \partial \mathcal{R} \end{aligned}$$

$$\Rightarrow \lambda > 0$$

Lahko razširimo X do $a \leadsto A_k = 0$

$$\sum_{k=1}^{\infty} X_k(x) = \sum B_k \sin\left(\frac{k\pi}{b-a} x\right) = g(x)$$

$$\leadsto B_k = \frac{2}{b-a} \int_a^b g(y) \sin\left(\frac{k\pi}{b-a} y\right)$$

$$u(t, x) = \sum_{k=1}^{\infty} e^{-\left(\frac{k\pi}{b-a}\right)^2 t} \sin\left(\frac{k\pi}{b-a} y\right) \cdot B_k$$

A1: $u(x,t) = \sum_{k=1}^{\infty} e^{-\mu_k^2 t} B_k \sin(\mu_k x)$ konvergira

Izrek:

• $I = [a, b] \subseteq \mathbb{R}$ • $V_k \in C^1(I)$

• $\exists \sigma_0 \in I$ • $\sum_{k=0}^{\infty} V_k(\sigma_0) < \infty$

• $\sum_{k=0}^{\infty} V_k'(\sigma)$ enakomerno konvergira na I

Potem $\frac{\partial}{\partial \sigma} \sum_{k=0}^{\infty} V_k(\sigma) = \sum_{k=0}^{\infty} \frac{\partial}{\partial \sigma} V_k(\sigma)$

Trditvi:

• $\sum_{k=1}^{\infty} |C_k| < \infty$

Potem za $u(t,x) = \sum_{k=0}^{\infty} C_k e^{-k^2 t} \sin(kx)$ velja

1) konvergira absolutno na $[0, \pi] \times [0, \infty) = \bar{\Omega}$

2) na $\bar{\Omega}$ določa zvezno funkcijo

3) $Lu = 0$ za $L = \partial_t - \partial_{xx}$

Dokaz:

1) in 2) $|u(t,x)| \leq \sum_{k=1}^{\infty} |C_k|$. Po Weierstrassovem

M-testu velja, da $u(t,x)$ enakomerno konvergira torej je zvezna.

3) $se^{-s} \leq \frac{1}{e}$ za $\forall s \geq 0$ (maksimum funkcije)

$|u_t| = \left| \sum B_k k^2 e^{-k^2 t} \sin(kx) \right| \leq \sum B_k \frac{1}{e} \frac{1}{e} \sin(kx) \leq \sum B_k$

$|u_{xx}| = \left| \sum B_k e^{-k^2 t} k^2 \sin(kx) \right| = |u_t| \leq \sum |B_k|$

$u_t = u_{xx} \Rightarrow Lu = 0$

Torej moramo pokazati, da je $\sum |B_k| < \infty$

$$B_k = \int_0^\pi g(y) \sin(ky) dy \leftarrow \begin{matrix} a=0 \\ b=\pi \end{matrix} \text{ vpeljava nove spremenljivke}$$

$$B_k = \langle g, \sin(k \cdot) \rangle = \|\sin(k \cdot)\| \langle g, \frac{\sin(k \cdot)}{\|\sin(k \cdot)\|} \rangle$$

$$\sum B_k = \sum \frac{2}{\pi} \langle g, \varphi_k \rangle \leq \|g\| \cdot \frac{2}{\pi}, \text{ kjer je } \varphi_k \text{ ONS}$$

Katere g lahko razvijemo v Fourierjevo vrsto

$$g(a) = g(b) = 0 \Rightarrow g \text{ lahko razširimo na } [2a-b, b]$$

Če je $g \in C^1$ potem Fourierjeva vrsta konvergira po točkah.

Nehomogena toplotna enačba

$$u_t - u_{xx} = f \quad u = 0 \text{ na } \partial U \times [0, \infty) \quad u = g \text{ na } U \times \{0\}$$

$$f(x, t) = \sum_{k=1}^{\infty} d_k(t) \sin\left(\frac{k\pi}{b-a} x\right) \quad \text{fourierjeva vrsta po } x,$$

da so zato lahko odvisni od t

$$\Rightarrow \text{Nastavimo } u(t, x) = \sum_{k=0}^{\infty} T_k(t) \sin(\mu_k x)$$

$$\Rightarrow T'_k - \mu_k^2 T_k = d_k \Rightarrow T_k(t) = e^{-\mu_k^2 t} D_k(t), \text{ kjer je}$$

$$D_k(t) = \int_0^t d_k(\tau) e^{\mu_k^2 \tau} d\tau + D_k(0)$$

$$u(x, 0) = \sum T_k(0) \sin(\mu_k x) = g(x) = \sum B_k \sin(\mu_k x)$$

$$T_k(0) = D_k(0) = B_k = \frac{2}{b-a} \int_a^b g(y) \sin(\mu_k y) dy$$

$$\Rightarrow u(x, t) = \sum e^{-\mu_k^2 t} \left(\int_0^t d_k(\tau) e^{\mu_k^2 \tau} d\tau + B_k \right) \sin(\mu_k x)$$

Ali je ta rešitev enolična?

$$v = u_1 - u_2, \quad v_t - v_{xx} = 0, \quad v = 0 \text{ na } \partial U \times [0, \infty), \quad v = 0 \text{ na } U \times \{0\}$$

$$E(t) = \int_U v^2 dV, \quad E'(t) = 2 \int_U v_t v dV = 2 \int_U v_x v_x dV - 2 \int_U v_x^2 dV \leq 0$$

$$E(0) = 0 \Rightarrow E \equiv 0 \Rightarrow v \equiv 0$$

HARMONIČNE FUNKCIJE

$$\Delta u = 0$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

$$\Delta = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$$

Verižno pravilo:

$$\frac{\partial}{\partial \bar{z}} (g \circ f) = \left(\frac{\partial}{\partial w} g \right) (f) \cdot \frac{\partial}{\partial \bar{z}} f + \left(\frac{\partial}{\partial \bar{w}} g \right) (f) \cdot \left(\frac{\partial}{\partial \bar{z}} f \right)$$

f holomorfna $\Leftrightarrow u, v$ zadošča Cauchy-Riemannovim

$$\text{systemu: } u_x = v_y \quad u_y = -v_x$$

$$\Leftrightarrow \frac{\partial}{\partial \bar{z}} f = 0$$

f holomorfna $\Rightarrow u, v$ harmonični

Izreki: $u \in C^2(\Omega)$

Polem u harmonična $\Leftrightarrow u \in L^pV$

Dokaz: $u \in L^pV \Leftrightarrow \varphi: r \mapsto \langle u \rangle_k(z_0, r) = u(z_0)$ konstantna

$$\begin{aligned} \varphi'(s) &= \int_0^{2\pi} \frac{d}{ds} u(z_0 + \cos \vartheta, z_0 + i \sin \vartheta) d\vartheta = \\ &= \int_0^{2\pi} (u_x, u_y) \cdot (\sin \vartheta, \cos \vartheta) d\vartheta = \int_{\partial k} \nabla u \cdot d\vec{s} = \\ &= \int_k \Delta u \, d\mathcal{S} \end{aligned}$$

Torej $u \in L^pV \Rightarrow \int_k \Delta u \, d\mathcal{S} = 0 \xrightarrow{s \rightarrow 0} \Delta u(z_0) = 0$

in $\Delta u = 0 \Rightarrow \varphi'(s) = 0 \Rightarrow \lim_{r \rightarrow 0} \varphi(r) = \text{konst} = u(z_0)$

$$\Rightarrow \varphi(r) = z_0$$

□

u subharmonična ($\Delta u \geq 0$) $\Rightarrow u(a) \leq \langle u \rangle_{\partial k(a, r)}$

Trditev:

- $\mathcal{D} \subset \mathbb{R}^2$ enostavno povezana
- u realna, harmonična na $\mathcal{D}$

Potem $\exists$ hol. funkcija f , da je $\operatorname{Re}(f) = u$

Dokaz:

Iščemo v , da je $\nabla v = \operatorname{grad}(v) = (-u_x, u_y)$

Torej želimo, da je $(-u_y, u_x)$ potencialno polje.

Vemo: $\operatorname{Rot} F = 0 \Rightarrow F$ potencialno

$$\operatorname{Rot}(-u_y, u_x) = u_{xx} + u_{yy} = 0 \Rightarrow \exists v, \operatorname{grad}(v) = (-u_x, u_y)$$

Lema: u harmonična $\wedge F$ holomorfna

$\Rightarrow u \circ F$ harmonična

Dokaz: Vemo: g holomorfna $\Rightarrow \operatorname{Re}(g)$ harmonična

$$u \circ F = (\operatorname{Re} f) \circ F = \operatorname{Re}(f \circ F)$$

$\uparrow$ holomorfna $\uparrow$ $u \circ F \in \mathbb{R}$ $\uparrow$ holomorfna

DIRICHLETOV PROBLEM NA $\mathbb{R}^2$

Iščemo $u: \bar{D} \rightarrow \mathbb{R}$

$u|_{\partial D} = f$, $u|_D$ harmonična, u zvezna

Za z^k ozirama $e^{2\pi i k}$ $k \in \mathbb{Z}$ že znamo razširiti
 $\leadsto$ bomo imeli za $\sum_{k=-\infty}^{\infty} a_k z^k$ $\downarrow$ $re^{2\pi i k t}$

Izpeljava Poissonovega jedra:

$$f = \sum a_k z^k \quad a_k = \langle f, e^{2\pi i k t} \rangle = \int_0^1 f(t) e^{2\pi i k t} dt$$

$$F = \sum a_k z^k \text{ na } \bar{D} : F = \sum r^{|k|} e^{2\pi i k t} \int_0^1 f(s) e^{2\pi i k s} ds \\ = \sum \int_0^1 f(s) e^{2\pi i k(t+s)} ds = \int_0^1 f(s) \sum_{k \in \mathbb{Z}} r^{|k|} e^{2\pi i k(t+s)} ds$$

Ali $\sum_{k \in \mathbb{Z}} r^{|k|} e^{2\pi i k(t+s)}$ konvergira enakomerno?

$$\sum_{k=0}^{\infty} (r e^{2\pi i(t+s)})^k + \sum_{k=1}^{\infty} (r e^{-2\pi i(t+s)})^k = \dots = \frac{1-r^2}{|1 - r e^{-2\pi i s}|^2} = \frac{1-|z|^2}{|1-z|^2} = \\ = \frac{1-r^2}{1-2r \cos(2\pi s) + r^2} = P_r(s) \text{ Poissonovo jedro}$$

Približna enota za $r \nearrow 1$

1. $\int_{\partial D} P_r(t) dt = 1$ za $\forall r \in [0, 1)$
2. $\int |P_r(t)| dt < c$ za $\forall r \in [0, 1)$
3. $\lim_{r \nearrow 1} \int_{\{x < t < \frac{1}{2}\}} |P_r(t)| dt = 0$ za $\forall \delta \in (0, \frac{1}{2})$

Izrek:

• $\{P_r(t), r \in [0, 1)\}$ približna enota za $r \nearrow 1$

Potem za $\forall f: \partial D \rightarrow \mathbb{R}$, $f * P_r \xrightarrow{r \nearrow 1} f$ enakomerno na ∂D . ($\lim_{r \nearrow 1} \max_{x \in \partial D} |(f * P_r)(x) - f(x)| = 0$)

Dokaz:

$$\begin{aligned} (f * P_r)(e^{2\pi i t}) - f(e^{2\pi i t}) &= \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t-s) P_r(s) ds - f(t) \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} P_r(s) ds = \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} P_r(s) (f(t-s) - f(t)) ds = \\ &\quad \int_{|s| < \delta} P_r(s) (f(t-s) - f(t)) ds + \int_{|s| \geq \delta} P_r(s) (f(t-s) - f(t)) ds \\ &= I + II \end{aligned}$$

I: f zvezna na $[-\frac{1}{2}, \frac{1}{2}] \Rightarrow$ enakomerno zvezna

$\Rightarrow$ δ izberemo tak, da $|f(t-s) - f(t)| < \varepsilon$

$\Rightarrow |I| \leq \left| \int_{|s| < \delta} P_r(s) \varepsilon ds \right| \leq \varepsilon \cdot \int_{\partial D} P_r(s) ds = \varepsilon \cdot C$

II: $II \leq 2 \cdot \max_{\partial D} f \cdot \underbrace{\int_{|s| \geq \delta} P_r(s) ds}_{< \varepsilon \text{ ker gre proti 0 za } r \nearrow 1} \leq 2 \max_{\partial D} f \cdot \varepsilon \quad \leftarrow \text{za } r \in (r_0, 1)$

$\Rightarrow$ Za primeren δ je $\max |(f * P_r) - f| < \varepsilon (1 + 2 \max f)$

$\Rightarrow f * P_r \xrightarrow{r \nearrow 1} f$ enakomerno na ∂D

□

Posledica: $r e^{2\pi i t} \mapsto \begin{cases} f(e^{2\pi i t}); & r=1 \\ (f * P_r)(e^{2\pi i t}); & r < 1 \end{cases}$ je zvezna

Izreki:

Poissonovo jedro $\frac{1-r^2}{1-2r\cos(2\pi t)+r^2}$ je približna enota

Dokaz:

$$\text{Velja: } \int P_r(t) dt = \frac{1}{\pi} \arctan\left(\frac{1+r}{1-r} \cdot \tan(\pi t)\right)$$

Lastnosti približne enote sledijo iz tega
in dejstva $P_r(t) > 0$ $\square$

Sklepamo $u = f * P_r$ reši Dirichletov problem

$$\begin{aligned} \Delta u &= \Delta \int f(t-s) P_r(s) ds = \int \Delta \left(f(s) \sum_{k \in \mathbb{Z}} r^{|k|} e^{2\pi i k(t-s)} \right) ds \\ &\quad \text{enakomerna konvergenca (zaradi } \mathbb{D} \text{ kompaktni)} \quad \text{neodvisno od } t \\ &= \int_0^1 f(s) \Delta \sum \dots ds = \int_0^1 f(s) \sum_{k \neq 0} \Delta (z^k + \bar{z}^k) w^{-1} dw = 0 \\ &\quad \text{enakomerna konvergenca } \sum \dots \end{aligned}$$

Želimo še pokazati enoličnost rešitve

Izrek: Princip maksima

• $\Omega \subseteq \mathbb{R}^2$ domojče • $u \in C^2$ • $u \in L^p V$

Potem če u doseže maksimum na $\Omega \Rightarrow u \equiv \text{const.}$

Dokaz:

Recimo, da u doseže maksimum pri z_0

$$\forall r < R, \quad u(z_0) = \langle u \rangle_{k(z_0, r)} \xrightarrow{\text{zmenost } u} u \equiv u(z_0) \text{ na } k(z_0, r)$$

Ker je to za $\forall r < R$ je $u \equiv u(z_0)$ na $k(z_0, R)$.

Torej je $\{u = u(z_0)\}$ odprta množica in zaprta,
ker je prazna točka (ki je zaprta množica). $\square$

Izrek: Dirichletov problem ima največ eno rešitev.

Dokaz:

u_1, u_2 rešitvi. $v = u_1 - u_2$

$\Delta v = 0 \Rightarrow v \in \text{LPV} \Rightarrow$ velja princip maksima.

$v|_{\partial D} = 0 \Rightarrow v \equiv 0 \Rightarrow u_1 = u_2$

□

Izrek:

• $u \in \mathcal{E}_1$

Potem $u \in \text{LPV} \Leftrightarrow u$ harmonična

Dokaz:

Na $K = K(z_0, R)$. $u|_K$ razširimo na $\tilde{u}$ harm. funkcijo na K .

Potem $u = \tilde{u}$ (po principu maksima) □

Harmonične konjugiranki al nebi

$$\partial \bar{\partial} f \rightarrow \bar{\partial} u \rightarrow \partial v \rightarrow \partial \bar{\partial} \bar{f}$$

$$F(z) = \frac{1+z}{1-z} \quad \operatorname{Re} F = P_r(z) = \frac{1-|z|^2}{|1-z|^2}$$

$$\text{Definirajmo } Q = \operatorname{Im} F = \frac{2 \operatorname{Im} z}{|1-z|^2} = \frac{2r \sin(2\pi t)}{1-2r \cos(2\pi t) + r^2}$$

Q_r ni približna enota

$$\text{Želimo: } v = f * Q_r \quad u = f * P_r$$

$$\begin{aligned} u + iv &= \int_0^1 (f(s)P(t-s) + if(s)Q(t-s)) ds = \\ &= \int_0^1 f(s)F(t-s) ds = f * F \end{aligned}$$

$$\begin{aligned} f * F &\text{ je holomorfa, ker } \frac{\partial}{\partial \bar{z}} \int f(w)F(zw^{-1}) dw = \\ &= \int f(w) \frac{\partial}{\partial \bar{z}} F(zw^{-1}) dw \underset{\text{kompozitum holomorfnih}}{=} \int f(w) \cdot 0 dw = 0 \end{aligned}$$

Torej mora $f * F$ enakomerno konvergirati za $\forall r$
To velja za $\forall r < 1$

$$\text{Želimo najti } Q_1, \text{ ker bo } g = f * Q_1$$

$$Q_1(t) := \lim_{r \nearrow 1} P_r(t) = \begin{cases} 0 & t=0 \\ \cot(\pi t) & t \neq 0 \end{cases} \quad \text{ker } \frac{\sin x}{1-\cos x} = \cot \frac{x}{2}$$

Izrek:

• f odvedljiva

Potem $\lim_{\delta \searrow 0} \int_{\{\delta < s < \frac{1}{2}\}} f(t-s) \cot(\pi s) ds$ obstaja

Dohaz:

$$\begin{aligned} \int_{\{\delta < s < \frac{1}{2}\}} f(t-s) \cot(\pi s) ds &= \int_{\delta}^{\frac{1}{2}} f(t-s) \cot(\pi s) - \int_{\delta}^{\frac{1}{2}} f(t+s) \cot(\pi s) ds \\ &= \int_{\delta}^{\frac{1}{2}} \cot(\pi s) (f(t-s) - f(t+s)) ds = \\ &= \int_{\delta}^{\frac{1}{2}} \cot(\pi s) \cdot s \left(\frac{f(t-s) - f(t)}{s} - \frac{f(t+s) - f(t)}{s} \right) ds \\ &= \int_{\delta}^{\frac{1}{2}} \underbrace{\cot(\pi s)}_{\text{obstaja v 0}} \cdot s \underbrace{(-f'(\xi_s) - f'(\eta_s))}_{\text{obstaja v 0}} ds \end{aligned}$$

Hilbertova transformiranka $H_f(t) = \lim_{\delta \searrow 0} \int_{\{\delta < s < \frac{1}{2}\}} f(t-s) \cot(\pi s) ds$

$$\text{Trditew: } \lim_{r \nearrow 1} (f * Q_r)(t) = H_f(t)$$

$$\text{Dokaz: } \delta = 1-r$$

$$\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} Q_r(s) f(t-s) ds = \int_{\{s < \delta\}} Q_r(s) f(t-s) ds + \int_{\{s \geq \delta\}} Q_r(s) f(t-s) ds \\ = I + II$$

$$I: f(t) \int_{\{s < \delta\}} Q_r(s) ds = 0 \quad (f \text{ host})$$

$$I = \int_{\{s < \delta\}} Q_r(s) (f(t-s) - f(t)) ds = \int_{\{s < \delta\}} Q_r(s) s \frac{f(t-s) - f(t)}{s} ds \leq \overbrace{\delta C_+}^{LC_+}$$

$$\leq 2\delta \cdot C_+ \cdot \sup(s Q_r(s))$$

$$s Q_r(s) = w \frac{\operatorname{Im} w}{|1-w|^2} = \frac{2s \sin s}{|1-w|^2} \leq \frac{2 \sin^2 s}{|1-w|^2} =$$

$$= \left(\sqrt{2} \frac{\operatorname{Im} w}{|1-w|} \right)^2 = \left(\sqrt{2} \frac{|w-\bar{w}|}{|1-w|} \right)^2 =$$

$$2 \left| \frac{1-w}{|1-w|} - \frac{1-\bar{w}}{|1-\bar{w}|} \right|^2 \leq 4$$

$$\Rightarrow I \leq 8 \delta C_+ \xrightarrow{\delta \searrow 0} 0$$

$$\text{Želimo torej } \lim_{\delta \searrow 0} II = H_f(t)$$

$$II - H_f(s) = \int_{\{s \leq \frac{1}{2}\}} (Q_1(s) - Q_r(s)) f'(t-s) ds$$

$$Q_1(s) - Q_r(s) = \frac{\sin(2s)}{1-2\cos(2s)} - \frac{2r \sin(2s)}{1-2r \cos(2s) + r^2} =$$

$$= \sin x \frac{1-2r \cos x + r^2 - 2r + 2r \cos x}{(1-\cos x)(1-2r \cos x + r^2)} = \frac{\sin x \cdot (1-2r+r^2)}{\dots}$$

$$= (1-r)^2 \frac{\sin x}{1-\cos^2 x} (1+\cos x) \frac{1}{1-2r \cos x + r^2} =$$

$$= (1-r)^2 \frac{1+\cos x}{\sin x} P_r(s) \frac{1}{1-r^2} = \frac{1-r}{1+r} \frac{1+\cos x}{\sin x} P_r(s)$$

$$\Rightarrow \left| \lim_{r \nearrow 1} II - H_f(s) \right| = 0$$

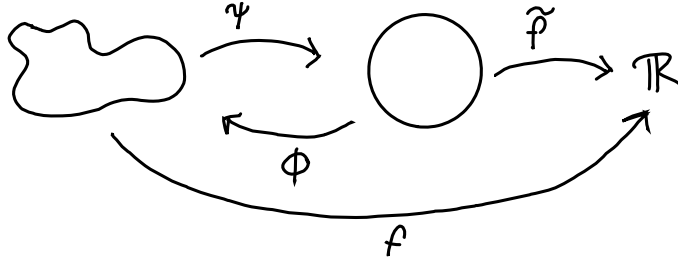
□

Tako smo našli našo konjugiranko α f

$$g = H_f(t) = \lim_{r \nearrow 1} (f * Q_r)(t)$$

DIRICHLETOV PROBLEM NA POLJUBNIH DOMENAH

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• imamo f na $\partial\Omega$

$$\bullet \quad f(x) = \tilde{f}(\psi(x))$$

$$\bullet \quad u(x) = \tilde{u}(\psi(x)) = (\tilde{f} * P_r) \psi(x)$$

$$\begin{aligned} u(z) &= u(\phi(\tilde{z})) = (u \circ \phi)(\tilde{z}) = \frac{1}{2\pi} \int_{\partial D} (f \circ \phi)(\tilde{w}) P(z, \tilde{w}) d\tilde{w} = \\ &= \int_{\partial D} f(\phi(\tilde{w})) P(z, \tilde{w}) d\tilde{w} \underset{w=\phi(\tilde{w})}{=} \frac{1}{2\pi} \int_{\partial \Omega} f(w) P(z, \overline{\psi(w)}) |d\psi(w)| dw \end{aligned}$$

$$P_{\Omega}(z, w) = \frac{1}{2\pi} P_r(z, \overline{\psi(w)}) |d\psi(w)|$$

$$\text{Torej imamo rešitev } u(z) = \int_{\partial \Omega} f(w) P_{\Omega}(z, w) dw$$

GREENOVE FUNKCIJE

Želimo rešiti $Lu = f$ ($L = \sum_{i=1}^n \frac{\partial^{m_i}}{\partial x_i^{m_i}}$)

$$Lw = \delta \quad / * f$$

$$L(w * f) = (Lw) * f = \delta * f = f$$

Torej če za L znamo rešiti $Lw = \delta$, bomo
mali rešiti $Lu = f$

Želimo odgovor za $L = \Delta$

$$\Delta u = f \quad \text{na } \Omega$$

$$u = g \quad \text{na } \partial\Omega$$

$$u \text{ zvezna na } \overline{\Omega}$$

Nastavek za w : $w(z) = \psi(|x-z|) = \psi(r)$

$$\Delta \psi = \psi'' + \frac{n-1}{r} \psi'$$

$$\psi' = \eta \quad \eta' = -\frac{n-1}{r} \eta$$

$$\eta^{-1} d\eta = -(n-1) r^{-1} dr$$

$$\ln \eta = (1-n) \ln r + C$$

$$\eta = C r^{1-n} \quad \text{tj. } \psi(r) = \begin{cases} C \ln r + D & ; n=2 \\ \frac{C}{2-n} r^{2-n} + D & ; n>2 \end{cases}$$

Zato za $n=2$ vzememo $K(z, \xi) = \frac{1}{2\pi} \log |z-\xi|$

Izrek: Greenova reprezentacijska formula

• $u \in C^2(\bar{M})$ • $z \in M$

Velja:

$$u(z) = \int_{\partial M} (u \cdot \nabla K - \nabla u \cdot K)(z, \cdot) d\vec{s} + \iint_M (K \cdot \Delta u)(z, \cdot) dS$$

Dokaz:

• Vzemimo $\Omega = M - \overline{K(z, \varepsilon)}$

• Greenova 3. identiteta:

$$\begin{aligned} \int_{\partial \Omega} (u \nabla K - \nabla u \cdot K) d\vec{s} &= \iint_{\Omega} (\overset{0}{u \cdot \Delta K} - \Delta u \cdot K) dS = \\ &= - \iint_{\Omega} \Delta u \cdot K dS \xrightarrow{\varepsilon \rightarrow 0} \int_M K \cdot \Delta u dS \end{aligned}$$

$$\Rightarrow \int_M K \cdot \Delta u dS = \int_{\partial M} (u \nabla K - \nabla u \cdot K) - \lim_{\varepsilon \rightarrow 0} \int_{K(z, \varepsilon)} (u \nabla K - \nabla u \cdot K) d\vec{s}$$

$$\int_{\partial K(z, \varepsilon)} \nabla u \cdot K dS = \frac{1}{2\pi} \int_{\partial K} \nabla u \cdot \log \varepsilon d\vec{s} \leq \frac{1}{2\pi} \log \varepsilon \max \nabla u \cdot 2\pi \varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0$$

$$\int_{\partial K} u \cdot \nabla K d\vec{s}:$$

$$\begin{aligned} \nabla(\log|z-z_1|) &= \frac{1}{2} \nabla(\log((z_1-z_1)^2 + (z_2-z_2)^2)) = \\ &= \frac{1}{2} \frac{1}{|z-z_1|^2} (2(z_1-z_1), 2(z_2-z_2)) = \frac{1}{\varepsilon^2} (z-z) \end{aligned}$$

$$\Rightarrow \int_{\partial K} u \nabla K d\vec{s} = \frac{1}{\varepsilon^2} (z-z) \frac{1}{2\pi} \int_{\partial K} u d\vec{s} \rightarrow \langle u \rangle_{\partial K} \rightarrow u(z)$$

$$\Rightarrow \int_M K \Delta u dS = \int_{\partial M} (u \nabla K - \nabla u \cdot K) d\vec{s} - u(z) \quad \square$$

$$\Gamma(f)(z) = \iint_M f(z) \cdot K(z, z) dS(z)$$

$$\Gamma(\Delta u) = \iint_M \Delta u \cdot K = u - \int_{\partial M} (u \nabla K - \nabla u \cdot K) d\vec{s}$$

Če je $u \in C_c^2(\mathbb{R})$ potem je $u = \Delta u = 0$ na ∂M

$$\Rightarrow \Gamma(\Delta u) = u$$

Torej imamo rešitev na C_c^∞ za $\Delta u = f$

$$u = \Gamma(\Delta u) = \Gamma(f)$$

Imamo torej formulo

$$u(z) = \int_{\partial M} (u \cdot \nabla K - \nabla u \cdot K)(z, \cdot) d\vec{s} + \iint_M (K \cdot \Delta u)(z, \cdot) dS,$$

ampak ne vemo koliko je ∇u , torej želimo, da je $K=0$ na ∂M , da se znebimo tega člena

Poskus: $G = K + h$ za nek harmoničen h

Poglejmo za poljuben h :

$$\begin{aligned} & \int_{\partial M} (u \nabla G - \nabla u G) d\vec{s} + \iint_M G \cdot \Delta u dS = \\ &= \int_{\partial M} (u \nabla K - \nabla u K) d\vec{s} + \iint_M K \Delta u dS + \int_{\partial M} u \nabla h - \nabla u h + \iint_M h \Delta u dS = \\ &= u(z) + \iint_M (u \Delta h - \Delta u h) dS + \int_{\partial M} h \Delta u dS = u(z) \end{aligned}$$

$$h(z, z) = \int_{\partial M} K(w, z) P_M(z, w) dw = \frac{1}{2\pi} \int_{\partial M} \log |w - z| P_M(z, w) dw$$

$$u(z) = \int_{\partial M} u \nabla G d\vec{s} + \iint_M G \cdot \Delta u dS$$

$G = K + h$ je **greenova funkcija** domene M za Laplacov operator

Rešitev dirichletovega problema

• $\Delta u = f$ na M • $u = g$ na ∂M • u zvezna na $\bar{M}$

$$\Delta u = f, u = g \text{ na } \partial M \Rightarrow u = \int_{\partial M} g \cdot \nabla G d\vec{s} + \iint_M G \cdot f$$

Poissonovo jedro definiramo kot $P_{M,z}(z) = \frac{\partial}{\partial n} G(z, z)$
 ↑
 odvod po
 zunanji normali

SCHWARTZOVE FUNKCIJE

$$\rho_{\alpha, \beta}(f) = \sup |x^\alpha \cdot \partial^\beta f| = \sup \left| \prod_{i=1}^n x_i^{\alpha_i} \cdot \frac{\partial^\beta f}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}} \right|$$

$$\rho_{\alpha, \beta}(f+g) = \sup |x^\alpha \cdot (\partial^\beta f + \partial^\beta g)| \leq \rho_{\alpha, \beta}(f) + \rho_{\alpha, \beta}(g)$$

$$\rho_{\alpha, \beta}(\lambda f) = \lambda \cdot \rho_{\alpha, \beta}(f)$$

Schwartzov razred $\mathcal{S}(\mathbb{R}^n) = \{f \in \mathcal{C}(\mathbb{R}^n); \rho_{\alpha, \beta}(f) < \infty\}$

$$\mathcal{C}_c \subseteq \mathcal{S} \subseteq \mathcal{C}^\infty$$

Trditel: $f \in \mathcal{C}^\infty$

$$f \in \mathcal{S} \iff \forall \beta \in \mathbb{N}_0^n. \forall n \in \mathbb{N}. \sup (1+|x|)^n |\partial^\beta f| < \infty$$

Dokaz: (priblizno)

$$\sup (1+|x|)^N |\partial^\beta f| = \sup \left(\sum_{k=1}^N a_k |x|^k |\partial^\beta f| \right)$$

$$(\Rightarrow) \sup |x^\alpha \cdot \partial^\beta f| \leq \sup |x|^{\alpha_1} |\partial^{\beta_1} f| \leq \sup (1+|x|)^{|\alpha|+1} |\partial^\beta f| < \infty$$

$$(\Leftarrow) \sup \left(\sum a_k |x|^k |\partial^\beta f| \right) \leq C \cdot \sup |x|^N |\partial^\beta f| \leq C \cdot \rho_{(N, N, \dots, N), \beta}(f) < \infty$$

Definiramo: $f_k \longrightarrow f \vee \mathcal{S} \iff \forall \alpha, \beta. \rho_{\alpha, \beta}(f_k - f) \xrightarrow{k \rightarrow \infty} 0$

FOURIERJEVA TRANSFORMACIJA

$$\mathcal{F}: f \mapsto \hat{f} \quad \hat{f}(z) = \int_{\mathbb{R}^n} f(z) e^{-2\pi i \langle z, z \rangle} dz$$

Izrek:

$$\bullet f, g \in \mathcal{S}(\mathbb{R}^n) \quad \bullet y \in \mathbb{R}^n \quad \bullet \alpha \in \mathbb{N}_0^n$$

Potem velja

$$1) \|\hat{f}\|_\infty \leq \|f\|_1 \quad (\sup |\hat{f}| \leq \int |f|)$$

$$8) \widehat{\partial^\alpha f}(z) = (2\pi i z)^\alpha \hat{f}(z)$$

$$9) \partial^\alpha \hat{f} = \widehat{(-2\pi i x)^\alpha f}(z)$$

$$10) \hat{f} \in \mathcal{S}$$

$$11) \widehat{f * g} = \hat{f} \cdot \hat{g}$$

Dokaz:

1)

$$\|\hat{f}\|_\infty = \sup \left| \int f(z) e^{2\pi i \langle z, z \rangle} dz \right| \leq \sup \int |f(z)| dz = \sup \|f\|_1 = \|f\|_1$$

8) Indukcija po $|\alpha|$

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \frac{\partial}{\partial x_i} \partial^{\alpha'} f(z) e^{-2\pi i \langle z, z \rangle} dz &= \int \partial^{\alpha'} f e^{-2\pi i \langle z, z \rangle} \Big|_{\mathbb{R}^{n-1}} + \\ &+ \int_{\mathbb{R}^n} \partial^{\alpha'} f(z) \cdot 2\pi i z_i e^{-2\pi i \langle z, z \rangle} dz = \\ &= 2\pi i z_i \cdot (2\pi i z)^\alpha \hat{f}(z) = (2\pi i z)^\alpha \hat{f}(z) \end{aligned}$$

$$\text{Baza: } \alpha = (0, 0, \dots, 0): \hat{f}(z) = (2\pi i z)^0 \hat{f}(z)$$

$$\begin{aligned} 9) \partial^\alpha \hat{f} &= \frac{\partial}{\partial x_i} \partial^{\alpha'} \hat{f} = \frac{\partial}{\partial x_i} \widehat{(-2\pi i x)^\alpha f}(z) = \\ &= \frac{\partial}{\partial z_i} \int (-2\pi i x)^\alpha f(x) e^{-2\pi i \langle x, z \rangle} dx = \\ &= \int (-2\pi i x)^\alpha f(x) e^{-2\pi i \langle x, z \rangle} dx = \widehat{(-2\pi i x)^\alpha f}(z) \end{aligned}$$

$$\text{Baza: } \hat{f} = \hat{f}$$

$$\begin{aligned} 10) \rho_{\alpha, \beta}(\hat{f}) &= \sup x^\alpha \cdot \partial^\beta \int f(z) e^{-2\pi i \langle x, z \rangle} dz = \\ &= \sup x^\alpha \int (-2\pi i z)^\beta f(z) e^{-2\pi i \langle x, z \rangle} dz = \\ &= \max \left\{ \sup_{\mathbb{R}^n} \dots, \sup_{\mathbb{R}^n} \dots \right\} \\ &\leq \max \left\{ C, \int (-2\pi i z)^\beta (-2\pi i)^\alpha f(z) e^{-2\pi i \langle x, z \rangle} dz \right\} \leq \\ &= \max \left\{ C, \frac{1}{(2\pi i)^\alpha} \partial^{\alpha+\beta} f \right\} < \infty \end{aligned}$$

$$\begin{aligned} 11) \widehat{f * g} &= \iint f(z) g(z-z) dz \cdot e^{-2\pi i \langle x, z \rangle} dz = \\ &= \int dz \int f(z) g(z-z) e^{-2\pi i \langle x, z-z \rangle} e^{-2\pi i \langle x, z \rangle} dz = \\ &= \int \hat{g}(x) \cdot f(z) e^{-2\pi i \langle x, z \rangle} dz = \hat{g} \cdot \hat{f} \end{aligned}$$

$$\check{f}(x) = \hat{f}(-x) \quad \text{inverzna fourierova transformacija}$$

Izrek:

$$\bullet f, g \in \mathcal{S}(\mathbb{R}^n)$$

Velja:

$$1) \mathcal{F}\hat{g} = \hat{\mathcal{F}g}$$

$$2) (\hat{\hat{f}})^{\vee} = (\check{\check{f}})^{\wedge} = f$$

$$3) \langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle \quad \text{za} \quad \langle f, \psi \rangle = \int f \cdot \bar{\psi}$$

$$4) \|f\|_2 = \|\hat{f}\|_2 = \|\check{f}\|_2 \quad \text{Plancherelova identiteta}$$

$$5) \mathcal{F}g = \hat{\mathcal{F}\check{g}}$$

Izrek: $\mathcal{F}: \mathcal{S} \rightarrow \mathcal{S}$ je homeomorfizem

$$\text{Bijektivnost: } \mathcal{F}(f) = \hat{f} \quad \Leftrightarrow \quad g = \mathcal{F}(\check{g})$$

Zveznost: Ali: za poljubno zaporedje $(f)_k \rightarrow f$
velja $\mathcal{F}(f_k) \rightarrow \mathcal{F}(f)$?

Recimo da $f_k \rightarrow f$ (torej $\rho_{\alpha, \beta}(f_k - f) \rightarrow 0$)

$$g = f_k - f \quad \hat{g} = \widehat{f_k - f} = \hat{f}_k - \hat{f}$$

$$\rho_{\alpha, \beta}(\hat{g}) = \|x^\alpha \partial^\beta \hat{g}\|_\infty = \|x^\alpha (-2\pi i)^\beta \hat{g}_k(x)\|_\infty =$$

$$\leq \|x^\alpha (-2\pi i)^{|\beta|} \cdot \widehat{z^\beta g_k}(x)\|_\infty =$$

$$= \|(-2\pi i x)^\alpha (-2\pi i)^{\beta - \alpha} \widehat{z^\beta g_k}(x)\|_\infty =$$

$$= \|(-2\pi i)^{\beta - \alpha} \widehat{\partial^\alpha (z^\beta g_k)}(x)\|_\infty =$$

$$\leq (-2\pi i)^{\beta - \alpha} \|\partial^\alpha (z^\beta g_k)\|_1 \leq C_n \sum_{|\beta| \leq n+1} \rho_{\beta, \alpha}(x^\beta g_k) =$$

$$= C_n \sum \sum x^\beta \partial^\alpha g_k \leq C_n \sum \rho_{\alpha, \beta}(g_k) \xrightarrow{k \rightarrow \infty} 0$$

Zveznost inverza:

$$\mathcal{F}(f) = \hat{f} \quad \mathcal{F}^{-1}(f) = \check{f} = \hat{f}(-x) = \hat{f} \circ (x \mapsto -x)$$

kompozitum dveh zveznih funkcij je zvezna $\square$

Rieman-Lebesguova lema

$$\forall f \in L^1(\mathbb{R}). \lim_{|z| \rightarrow \infty} |\hat{f}(z)| = 0$$

Dokaz:

Vemo: $C_c^\infty \subseteq \mathcal{F} \subseteq L^1$ in C_c^∞ gosta v L^1

Zato je $\mathcal{F}$ gosta v L^1 (tudi v L^2)

$f \in L^1$ poljubno in $g \in \mathcal{F}$ tak, da velja $\|f - g\|_{L^1} < \varepsilon$

$$|\hat{f}(z)| \leq |\hat{f}(z) - \hat{g}(z)| + |\hat{g}(z)| \leq \widehat{|f - g|}(z) + \varepsilon \leq 2\varepsilon$$

redovlj vektor z ↓

$$\|f - g\|_{L^1} = \int |f - g| dz \geq \left| \int (f - g) e^{-2\pi i \langle x, z \rangle} dz \right| = \widehat{|f - g|}(z)$$

□

RAZŠIRITEV $\mathcal{F}$ NA L^p $1 < p < 2$

Želimo: $f = f_1 + f_2$; $f_1 \in L_1$, $f_2 \in L_2$

$$f_1 = f \cdot \chi_{\{|f| > 1\}}, \quad f_2 = f \cdot \chi_{\{|f| \leq 1\}} \Rightarrow f \leq f^p$$

Izrek: Hausdorff-Young

$$\bullet p \in [1, 2] \quad \bullet f \in L^p \quad \bullet \frac{1}{p} + \frac{1}{q} = 1$$

$$\text{Velja } \|\hat{f}\|_q \leq \|f\|_p$$

Izrek: Beckner

$$\bullet p \in [1, 2] \quad \bullet f \in L^p \quad \bullet \frac{1}{p} + \frac{1}{q} = 1$$

$$\text{Velja točna ocena } \|\hat{f}\|_q \leq \left(p^{\frac{1}{p}} \cdot q^{-\frac{1}{q}} \right)^{\frac{n}{2}} \|f\|_p$$

Ekstremalne so Gaussove funkcije

