

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}$$

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$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right) \quad \text{gradient}$$

$$\nabla \cdot F(x) = \frac{\partial F_1}{\partial x_1}(x) + \dots + \frac{\partial F_n}{\partial x_n}(x) \quad \text{divergence}$$

$$\nabla \times F(x) = \underset{n=3}{\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \dots + \frac{\partial^2 f}{\partial x_n^2}$$

$$n=3$$

$$\nabla \times \nabla f = 0$$

$$\nabla \times \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \dots, \dots \right) = \vec{0}$$

$$\nabla \cdot (\nabla \times F) = 0$$

$$\nabla \cdot (-y/z + z_y, z_x + x_z, -x_y + y_x) =$$

$$\cancel{z_x} - \cancel{y_z} + \cancel{x_z} - \cancel{z_x} + \cancel{y_z} - \cancel{x_y} = 0$$

cotomological complex, ke imaš dve

zaporedn: puščici je 0

$$\mathbb{R} \xrightarrow{\nabla} \mathbb{R}^3 \xrightarrow{\nabla \times} \mathbb{R}^3 \xrightarrow{\nabla \cdot} \mathbb{R}$$

$$K = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2$$

$$\int_K \nabla \cdot \vec{F} \, dx \, dy = \int_{\partial K} \vec{F} \, d\vec{n} \quad \text{normale}$$

$$\begin{aligned} \int_0^1 \int_0^1 \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) dx \, dy &= \int_0^1 dy \int_0^1 \frac{\partial F_1(x, y)}{\partial x} dx + \int_0^1 dx \int_0^1 \frac{\partial F_2(x, y)}{\partial y} dy \\ &= \int_0^1 F_1(x, y) \Big|_0^1 dy + \int_0^1 F_2(x, y) \Big|_0^1 dx = \end{aligned}$$

$$= \int_0^1 (F_1(1, y) - F_1(0, y)) dy + \int_0^1 (F_2(x, 1) - F_2(x, 0)) dx$$

$$\int_{\partial K} \vec{F} \, d\vec{n} = \int_0^1 (F_1(x, 0), F_2(x, 0)) \begin{pmatrix} dx \\ 0 \end{pmatrix} + \dots$$

$$= \int_0^1 -F_2(x, 0) dx + \dots = \int_0^1 (F_1(1, y) - F_1(0, y)) dy + \int_0^1 (F_2(x, 1) - F_2(x, 0)) dx$$

$$u_{xy} = x^2 y \quad u(x, 0) = x \quad u(0, y) = y^2$$

$$u_x = \frac{1}{2} x^2 y^2 + C(x)$$

$$u = \frac{1}{6} x^3 y^2 + C(x) + D(y)$$

$$u(x, 0) = C(x) + D(0) = x$$

$$u(0, y) = C(0) + D(y) = y^2$$

$$C(x) + D(y) = x + y^2$$

$$C(0) + D(0) = 0$$

$$u(x, y) = \frac{1}{6} x^3 y^2 + x + y^2$$

$$U_{xy} + U_x = y e^{-y}$$

$$u(0, y) = 0$$

$$u(x, x) = 0$$

integrando p.o. x

$$u_y(0, y) = 0$$

$$\frac{u(x, x)}{\partial y} = 0$$

$$U_y + U = x y e^{-y} + C(y)$$

$$v(0, y) = 0$$

$$0 + 0 = 0 + C(y) = 0$$

$$U_y + U = x y e^{-y}$$

lineal en diferenciales exactas

homogen del

$$u + u_y = 0$$

$$u = -\frac{\partial u}{\partial y} \quad dy = -\frac{du}{u} \Rightarrow y = -\ln u + C(x)$$

$$u = C e^{-y}$$

particular

$$C'(y) e^{-y} = x y e^{-y}$$

$$C'(y) = x y$$

$$C(x, y) = \frac{1}{2} x y^2$$

$$u = \frac{1}{2} x y^2 e^{-y} + C(x) e^{-y}$$

$$u(0, y) = C(0) e^{-y} = 0$$

$$u(x, x) = \frac{1}{2} x^3 e^{-x} + C(x) e^{-x} = 0$$

$$\Rightarrow C(x) = -\frac{1}{2} x^3$$

$$u(x, y) = \frac{1}{2} x e^{-x} (y^2 - x^2)$$

c)

$$u_x + u_y = 0$$

$$u(x, 0) = x^2$$

$$u(x, y) = \sum_{i,j} a_{ij} x^i y^j$$

$$u(x, 0) = \sum_i a_{i0} x^i = x^2 \Rightarrow a_{20} = 1$$

$$a_{i0} = 0 \quad i \neq 2$$

$$u_x = \sum_{i,j} a_{ij} i x^{i-1} y^j = a_{10} + \sum_{i,j \geq 0} a_{i+1,j} (i+1) x^i y^j$$

$$u_y = \sum_{i,j} a_{ij} j x^i y^{j-1} = a_{01} + \sum_{i,j \geq 0} a_{i,j+1} (j+1) x^i y^j$$

$$u_x + u_y = a_{01} + a_{10} + \sum_{i,j \geq 0} (a_{i+1,j} (i+1) + a_{i,j+1} (j+1)) x^i y^j = 0$$

$$\Rightarrow (a_{i+1,j} (i+1) + a_{i,j+1} (j+1)) = 0 \quad \forall x, y$$

$$a_{01} + a_{10} = 0 \Rightarrow a_{01} = -a_{10}$$

$$a_{2,1} = a_{1,2}$$

$$\Rightarrow a_{0j} = 0 \text{ razem } a_{02} = -2$$

$$a_{1,1} = a_{0,2}$$

$$a_{1,1} = -a_{2,0}$$

$$a_{1,0} + a_{0,1} = 0, a_{1,1} + a_{0,2} = 0 \Rightarrow a_{1,1} = 2$$

$j \backslash i$	0	1	2	3
0	a_{00}	0	1	0
1	0	-2	0	0
2	1	0	0	
3	0	0		

$$a_{1,1} = -a_{0,2}$$

$$a_{3,0} = -a_{2,1}$$

$$u(x, y) = a_{00} + a_{2,0} x^2 + a_{1,1} xy + a_{0,2} y^2$$

$$= a_{00} + x^2 - xy + y^2$$

$$u(x, 0) = a_{00} + x^2 \Rightarrow a_{00} = 0$$

$$\Rightarrow u(x, y) = x^2 - xy + y^2$$

$$a_{2,0} = a_{1,1}$$

Hmmmm nek: je narobe

$$u(x, y) = x^2 - 2xy + y^2$$

Pozabila sem

množititi z $i+1$ in

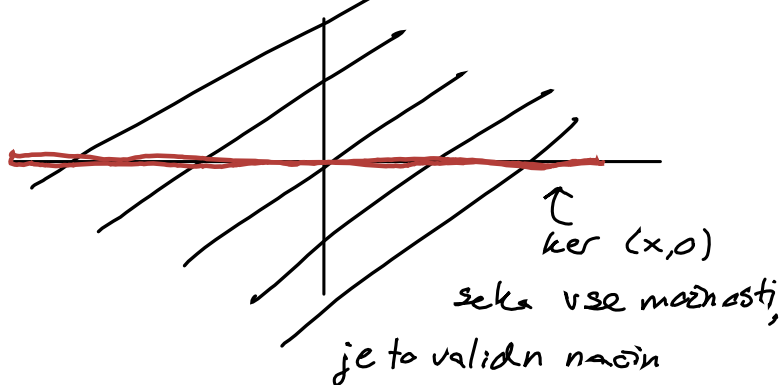
$j+1$ pri odvajanju

3. c)

$$u_x + u_y = 0 \quad u(x, 0) = x^2 \quad \leadsto u(x, y) = (x - y)^2$$

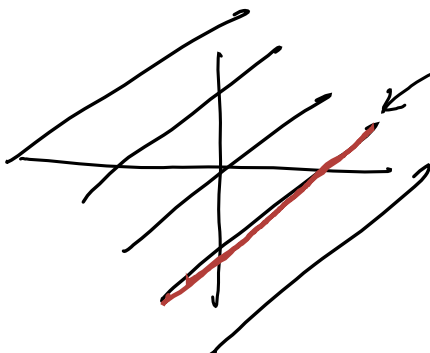
Naj bo $u(x, y) = g(x - y)$

$$\Rightarrow u_x + u_y = g'(x - y) - g'(x - y) = 0$$



Slab primer

$$u(x, x) = x^2$$



če imamo tak robni:
pogoji ne moremo
dobiti rešitve

Ali obstaja, ali pa jui
je veliko

$$u_{xy} + yu_y + u = 1$$

$$u_x(x, 0) = 1$$

$$u(x, x) = e^{-x^2}$$

$$yu_y + u = (yu)_y$$

$$u_{xy} + (yu)_y = 1 \leadsto u_x + yu = y + C(x)$$

$$\underset{0}{V}(x, 0): u_x(x, 0) + 0 \cdot 1 = 0 + C(x) \Rightarrow C(x) = 0$$

$$u_x + yu = y$$

homogeno:

$$u_x = -yu$$

$$\frac{\partial u}{u} = -y dx$$

$$u = D(y)e^{-yx}$$

partikularna:

$$D_x(x, y)e^{-yx} = y$$

$\leadsto$

$$D(x, y) = \int y e^{+yx} dx = e^{+yx} + C(y)$$

$$u = 1 + C(y)e^{-yx}$$

$$u(x, x) = 1 + C(x)e^{-x^2} = e^{-x^2} \rightarrow C(x) = \frac{e^{-x^2} - 1}{e^{-x^2}} = 1 - e^{x^2}$$

$$u(x, y) = 1 + (1 - e^{-y^2})e^{-yx}$$

$$u_{xy} + 2u_y = e^{-2x} \quad u(x,0) = e^{-x} \quad u(0,y) = 1$$

$$\begin{cases} u_x(x,0) = -e^{-x} \end{cases}$$

integriramo po y

$$\Rightarrow u_x + 2u = ye^{2x} + C(x)$$

$$v(x,0): -e^{-x} + 2e^{-x} = C(x) = e^{-x}$$

$$u_x + 2u = ye^{-2x} + e^{-x}$$

hom:

$$\frac{\partial u}{u} = -2dx \Rightarrow u = e^{-2x} C(y)$$

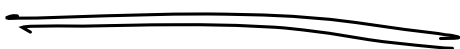
$$\text{part: } C_x(x,y)e^{-2x} = ye^{-2x} + e^{-x}$$

$$C(x,y) = \int (y + e^x) dx = yx + e^x + D(y)$$

$$u = yxe^{-2x} + e^{-x} + e^{-2x} D(y)$$

$$v(0,y): 0 + 1 + D(y) = 1 \Rightarrow D(y) = 0$$

$$\Rightarrow u = yxe^{-2x} + e^{-x}$$



KLE

4.3

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

$$S = \{ (x,y,z) \in \mathbb{R}^3 ; \quad z = u(x,y) \text{ u rešitev} \}$$

je integralne krivulje

Presek $S_1 \cap S_2$ je karakteristična krivulja

karakteristična krivulja $\gamma: \gamma(t) = (a(\gamma(t)), b(\gamma(t)), c(\gamma(t)))$

Izrek:

- S_1, S_2 integralni ploskvi za isto KLE. $p \in S_1 \cap S_2 \Rightarrow$ karakteristična krivulja, ki vsebuje p leži celo v $S_1 \cap S_2$
- $S_1 \cap S_2$, če se ne sekata tangento, je karakteristična krivulja

Dahet 2)

$$p \in C = S_1 \cap S_2$$

$$\pi_1 = T_p S_1 \quad \pi_2 = T_p S_2$$

$\dim(\pi_1 \cap \pi_2) = 1$, ker se ne sekata S_1 in S_2 tangento.

$\pi_1 \cap \pi_2$ tangenta na C

$$a(p), b(p), c(p) \in \pi_1 \cap \pi_2 = T_p C$$

$$T_p C = \{ \lambda (a(p), b(p), c(p)) + p ; \lambda \in \mathbb{R} \}$$

C je tangenta na karakteristično smer

$\Rightarrow C$ je karak. krivulja

①

$$x u_x + y u_y = 0 \quad (x, y, z) = (a, b, c)$$

$$\dot{x} = x \rightsquigarrow x_0(s) e^t$$

$$\dot{y} = y \rightsquigarrow y_0(s) e^t$$

$$\dot{z} = 0 \rightsquigarrow z = z_0(s)$$

$$\varphi(t) = u(tx, ty)$$

$$\dot{\varphi}(t) = u_x \cdot x + u_y \cdot y = 0$$

$$\Rightarrow \varphi(t) = C(s) = C \quad \varphi(0) = u(0, 0)$$

$$\Rightarrow \varphi(t) = u(0, 0)$$

$$\varphi(1) = u(x, y) = C = u(0, 0)$$

$$\Rightarrow u(x, y) = u(0, 0)$$

$$② \quad u: \bar{D} \rightarrow \mathbb{R}$$

$$a(x,y) u_x + b(x,y) u_y = -u$$

$$x a(x,y) + y b(x,y) > 0 \quad \text{za } x,y \in \partial D$$

$$\min, \max D \Rightarrow u \equiv 0$$

$$\max D \Rightarrow u_x = u_y = 0 \vee (x_0, y_0)$$

$$\Rightarrow -u(x_0, y_0) = 0$$

$$\min D \Rightarrow -u(x_1, y_1) = 0 \Rightarrow u \equiv 0$$

$$b) \quad (x,y,\lambda) \mapsto u(x,y) + \lambda(x^2 + y^2 - 1)$$

$$u_x(x,y) + 2\lambda x = 0$$

$$u_y + 2\lambda y = 0$$

$$x^2 + y^2 - 1 = 0$$

$$c) \quad t \mapsto u(tx_0, ty_0) \quad x_0, y_0 \text{ maks/min na robu}$$

$$\dot{f} = u_x x_0 + u_y y_0 \geq 0 \vee \text{dodici } t=1$$

$$t=1: u_x(x_0, y_0) x_0 + u_y(x_0, y_0) y_0 = -2\lambda(x_0^2 + y_0^2)$$

$$\Rightarrow \lambda \leq 0$$

$$\text{Dokazimo da } u(x_0, y_0) > 0$$

$$a(x,y) (-2\lambda x_0) + b(x,y) (-2\lambda y_0) \stackrel{=-u}{< 0}$$

$$(-2\lambda) \underbrace{(a(x,y)x_0 + b(x,y)y_0)}_{>0} < 0$$

$$\Rightarrow -2\lambda < 0 \Rightarrow \lambda > 0 \quad *$$

$$\text{Sledi } u(x_0, y_0) < 0$$

$$\text{Kjer keli je minimum uve kurti } \geq 0 \Rightarrow$$

$$u \equiv 0$$

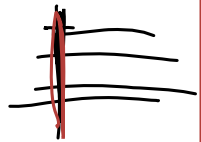
Metode karakteristik za enačbe 1. reda
enačba je

- $a(x, u(x)) \cdot Du = b(x, u(x))$
- $u(x) = g(x)$ začetni pogoji
na krivulji Γ $g: \Gamma \rightarrow \mathbb{R}$

① Rešimo sistem za vse y po s

od tod $\begin{cases} \dot{z}(y, s) = b(x(y, s), z(y, s)) \\ \dot{x}(y, s) = a(x(y, s), z(y, s)) \end{cases}$

po s $x(y, 0) = y$; $z(y, 0) = g(y)$



② Najdemo $s(x)$, $y(x)$, potem $u(x) = z(y(x), s(x))$

$$\textcircled{1} \quad u(0, x) = g(x)$$

$$u_t + a \cdot \nabla u = 0$$

$$\dot{z}(y, s) = 0 \quad \leadsto \quad z = C(y)$$

$$\dot{t}(y, s) = 1 \quad \leadsto \quad t = s + C(y)$$

$$\dot{x}(y, s) = a \quad \leadsto \quad x = as + C(y)$$

$$(t(y, 0), x(y, 0)) = (0, y)$$

$$z(y, 0) = g(y)$$

$$z = C(y) = g(y)$$

$$t(y, 0) = C(y) = 0 \leadsto t(y, s) = s$$

$$x(y, 0) = C(y) = y \leadsto x(y, s) = as + y$$

$$y = x - at$$

$$u(x, y) = g(x - at)$$

b)

$$u_t + a \triangleright u + u = f(x)$$

$$u_t + a u = f(x) - u$$

$$\dot{z}(y, s) = f(x(y, s)) - z(y, s)$$

$$\dot{t}(y, s) = 1 \leadsto t = s + C(y)$$

$$\dot{x}(y, s) = a \leadsto x = s + D(y)$$

$$x(y, 0) = g(y) \leadsto x = s + g(y)$$

$$t(y, 0) = 0 \leadsto t = s$$

$$\leadsto \dot{z}(y, s) = f(s + g(y)) - z$$

$$\text{hom: } \int \frac{dz}{z} = - \int ds \leadsto z = C(y) e^{-s}$$

$$\text{par: } \dot{z}(y, s) e^{-s} = f(s + g(y))$$

$$\leadsto C = \int f(s + g(y)) e^s ds$$

$$z(y, s) = \int f(t + g(y)) e^t dt e^{-s}$$

$$= \int f(t) e^t dt e^{-g(y)} e^{-s}$$

$$u(x, y) = \left(\int f(t) e^t dt \right) e^{-x}$$

$$c) u_t + \nabla F(u) = 0$$

$$\begin{aligned} \nabla F(u(x_1, \dots, x_n)) &= (F'(u) \frac{\partial u}{\partial x_1}, \dots, F'(u) \frac{\partial u}{\partial x_n}) = \\ &= F'(u) \cdot \nabla u \end{aligned}$$

$$\dot{z} = 0 \quad \leadsto \quad z = c(y) = g(y)$$

$$\dot{x} = F'(z) \leadsto \dot{x} = F'(c) \leadsto x = F'(g(y))s + y$$

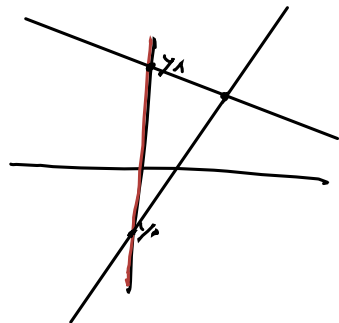
$$\dot{t} = 1 \quad \leadsto \quad t = s + D(y) \leadsto t = s$$

$$t(y, 0) = 0$$

$$x(y, 0) = y$$

$$z(y, 0) = g(y)$$

$$x(y, s) = F'(g(y))s + y$$



keaj ka se selete

$$F'(g(y))s + y_0 = F'(g(y_1))s + y_0$$

če se selete mora biti $g(y_0) = g(y_1)$

torej če se sedek, se ne more sedeti
amrelet resten abekaj's b'ekno

$$a) u u_x + u_y = 1 \quad u(x, x) = \frac{x}{2}$$

$$a = u \quad \dot{x} = z \rightarrow x = \frac{1}{2}t^2 + \frac{s}{2}t + C(s) \rightarrow$$

$$b = 1 \quad \dot{y} = 1 \rightarrow t + C(s) \rightarrow y = t + s$$

$$c = 1 \quad \dot{z} = 1 \rightarrow t + C(s) \rightarrow z = t + \frac{s}{2}$$

$$x(s, 0) = s$$

$$y(s, 0) = s$$

$$z(s, 0) = \frac{s}{2}$$

$$x = \frac{1}{2}t^2 + \frac{1}{2}st + s \rightarrow s = -$$

$$t = y - s \rightarrow s = y - t$$

$$u(x, y) = \frac{s}{2} + t = \frac{s}{2} + y - s = y - \frac{s}{2}$$

$$x = \frac{1}{2}t^2 + \frac{1}{2}y t - t^2 + y - t = -\frac{1}{2}t^2 + \frac{1}{2}y t + y - t$$

$$s = \frac{3x - y^2}{2 - y}$$

$$\begin{aligned} & \text{CC } x \neq 2 \text{ für } v \text{ reell} \\ & \text{und } \cancel{g_2} \neq 2 \end{aligned}$$

$$b) -\sin(y) u_x + x u_y = 0 \quad u(x,0) = g(x) \quad x > 0$$

$$\begin{cases} \dot{x} = -\sin(y) \\ \dot{y} = x \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} \ddot{y} = \dot{x} = -\sin(y) \\ \frac{d^2}{dt^2} \sin(y) = \frac{d}{dt} \cos y \cdot \dot{y} = \\ = \dot{y} \cos y + \dot{y} (-\sin y) \dot{y} \end{cases}$$

$$x(s,0) = x$$

$$y(s,0) = 0$$

$$z(s,0) = g(s)$$

1. energija $E_1(x,y,z) = z$

$$\ddot{y} = \dot{x} = -\sin(y)$$

2. energija

$$x \dot{x} = x \ddot{y} = \dot{y} \ddot{y} = -x \sin(y) = -\dot{y} \sin y$$

$$x \dot{x} = (\cos y)$$

$$\int x dx \int \cos y dt = \cos y + C$$

$$\frac{1}{2} x^2 = \cos y + C$$

$$E_2 = \frac{1}{2} x^2 - \cos y$$

Izračunaj H.

$$E_1(x(0), y(0), z(0)) = g(s)$$

$$E_2(x(s,0), y(s,0), z(s,0)) = \frac{1}{2} s^2 - 1$$

$$H(a,b) = a - g(\sqrt{2b+1})$$

$$H(E_1, E_2) = g(s) - g(\sqrt{s^2}) = 0$$

$$H(E_1(x,y,z), E_2(x,y,z)) = 0 \quad \forall x, y, z$$

||
z

$$z - g(\sqrt{x^2 - 2\cos y + 2}) = 0$$

$$\Rightarrow u(x,y) = g(\sqrt{x^2 - 2\cos y + 2})$$

$$c) \quad x(y^2+u)u_x - y(x^2+u)u_y = (x^2-y^2)u$$

$$u(x,y) = C \in \mathbb{R} \quad x \neq 0$$

$$\dot{x} = x(y^2+z)$$

$$\dot{y} = -y(x^2+z)$$

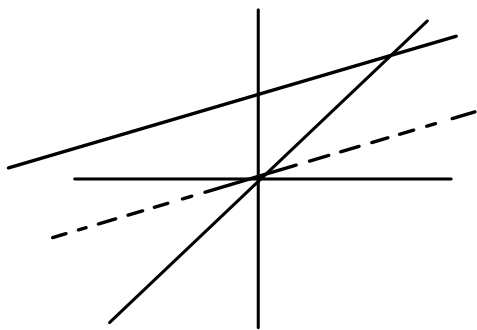
$$\dot{z} = z(x^2-y^2) = z(x-y)(x+y)$$

$$x(s,0) = s$$

$$y(s,0) = s$$

$$z(s,0) = c$$

$$\dot{x}(s,0) = s(s^2+c)$$



$$\dot{x}x + \dot{y}y = z(x^2-y^2) = \dot{z}$$

$$\frac{x^2}{2} + \frac{y^2}{2} = z + D(s)$$

$$E_1(x,y,z) = \frac{x^2}{2} + \frac{y^2}{2} - z$$

$$\frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z} = y^2 - x^2 + x^2 - y^2 = 0$$

$$\ln x + \ln y + \ln z = C$$

$$xyz = F(s) \quad E_2(x,y,z) = xyz$$

$$E_1(s,0) = s^2 - c = a$$

$$E_2(s,0) = cs^2 = b \quad ca - b + c^2$$

$$H(a,b) = ca - b + c^2$$

$$\nabla H(E_1, E_2) =$$

$$c\left(\frac{x^2}{2} + \frac{y^2}{2} - z\right) - xyz + c^2 = 0$$

is zero

$$z = u(x,y) = \frac{c^2 + \frac{c}{2}(x^2+y^2)}{c + xy}$$

Energija

Pri uporabi karakteristik v n -dimensionalnem sistemu, lahko iščemo n "energij". To so funkcije $E_j(x(t))$ za katere velja, da so ~~konstante~~

$$\frac{d}{dt} E_j(x(t)) = 0$$

Nato predpišemo $H(E_1, \dots, E_n) = 0$ (oz. poljubno konstanto) in z uporabo začetnih pogojev določimo H , ki zadošča $H(E_i(x(0)), \dots) = 0$

12.3

$$u_x + xu_y = u$$

$$\dot{x} = 1$$

$$\dot{y} = x$$

$$\dot{z} = z$$

$$u(1, y) = h(y) \quad h \in \mathcal{C}(\mathbb{R})$$

$$x(s, 0) = 1$$

$$y(s, 0) = s$$

$$z(s, 0) = h(y)$$

$$x = t + 1$$

$$y = \frac{1}{2}t^2 + t + s$$

$$z = h(y) e^{x-1}$$

$$\dot{z} = z$$

$$\frac{dz}{z} = dt$$

$$z = ce^t = h(y) e^{x-1}$$

$$t = x - 1$$

$$s = y - \frac{1}{2}(x-1)^2 - (x-1)$$

$$u(x, y) = h\left(y - \frac{1}{2}(x-1)^2 - (x-1)\right) e^{x-1}$$

$$x^2 u_x + y^2 u_y = (x+y)u \quad u(x, 1) = 1$$

$$x(s, 0) = s$$

$$\dot{x} = x^2 \leadsto -\frac{1}{x} = t + C \leadsto x = \frac{-1}{t+C} \leadsto x = \frac{s}{1-ts}$$

$$\dot{y} = y^2 \leadsto y = \dots = \frac{1}{1-t}$$

$$\dot{z} = (x+y)z$$

$$\frac{\dot{z}}{z} = \frac{1}{1-t} + \frac{s}{1-ts}$$

$$\ln z = -\ln(1-t) - \ln(1-ts) +$$

nakej priede
se zezeh.
megaj

$$z = \cancel{\frac{1-t+s}{1-t}} \frac{1}{(1-t)(1-ts)}$$

$$z = y \cdot \frac{x}{s}$$

$$s = x(1-ts)$$

$$s(1+tx) = x \leadsto s = \frac{x}{1-tx}$$

$$t = \frac{x-1}{y}$$

$$(1-t)y = 1$$

$$t = -\frac{1}{y} + 1 = 1 - \frac{1}{y} = \frac{y-1}{y}$$

$$s = \frac{x}{1 - \frac{(y-1)x}{y}} = \frac{xy}{y+x-xy}$$

$$z = y + x - xy \leadsto u(x, y) = y + x - xy$$

$$u(X, 0) = 1$$

$$x = \frac{1}{1+c}$$

$$y = \frac{1}{1+c}$$

$$x(s, 0) = s$$

$$y(s, 0) = 0$$

$$z(s, 0) = 1$$

$$\frac{1}{c} = s \Rightarrow \frac{1}{c} = \frac{1}{s} \Rightarrow x = \frac{s}{1+s}$$

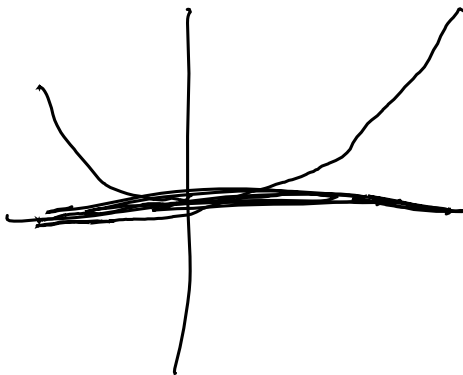
$$\frac{1}{c} = 0 \Rightarrow c = \infty \quad y = 0$$

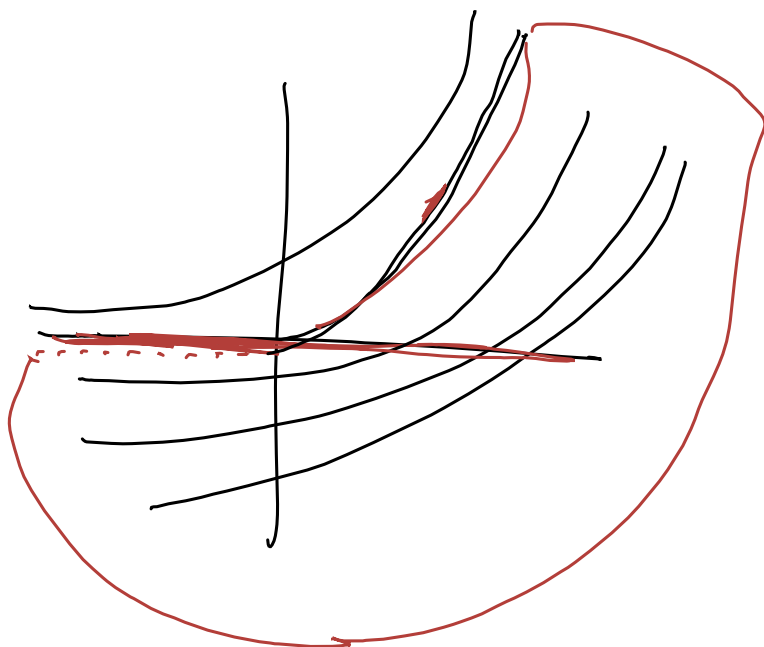
$$\dot{z} = \frac{s}{1+s} z$$

$$\ln z = -\ln(1+s) + C \Rightarrow$$

$$z = \frac{E(s)}{1+s} = \frac{1}{1+s} \quad \text{ne moremo izraziti}$$

$\Rightarrow x$ in y $\Rightarrow$ ne dajemo checks





Ogrinjača družine ravnin skozi točko

Naj bo $\Pi_\lambda = \{ (x, y, z) \mid z = \psi(x, y, \lambda) = a(\lambda)x + b(\lambda)y \}$
 družina ravnin skozi izhodišče
 $a, b \in C(I)$ $I \subseteq \mathbb{R}$ interval, da je
 $\vec{n}(\lambda) = (a(\lambda), b(\lambda), -1)$ injektivna
 in $w(a', b') = \begin{vmatrix} a' & b' \\ a'' & b'' \end{vmatrix} \neq 0$

Opozimo: $z = -1$ komponenta je normalizacija,
 ki da $\Pi_{\lambda_1} = \Pi_{\lambda_2} \Leftrightarrow \lambda_1 = \lambda_2$

Cilj: Najdemo $\varphi(x, y)$, ki definira $\sigma = \{ (x, y, \varphi(x, y)) \}$
 da velja

- $\sigma \subseteq \bigcup_\lambda \Pi_\lambda$
- $p_0 = (x_0, y_0, z_0) \in \sigma \cap \Pi_\lambda$ za nek λ
 $\Rightarrow$ normalni vektor za σ v točki p_0
 je sorazmeren z $\vec{n}(\lambda)$

Opozka: $\sigma \subseteq \bigcup_{\lambda \in I} \Pi_\lambda$ za fiksen λ je trivialna
 rešitev, ki jo ignoriramo

ker je $\vec{n}(\lambda)$ injektivna, če $p_0 \in \sigma \cap \Pi_\lambda$ in $p_0 \in \sigma \cap \Pi_\mu$
 potem $\lambda = \mu$

Consider the map $P: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

če $(x, y) = P(x, y, z)$ za nek $(x, y, z) \in \sigma$
 potem $p \in \Pi_\lambda$ za nek unikatni $\lambda \in I(x, y)$

$$\lambda: P(\sigma) \rightarrow \mathbb{R}$$

$$(x, y) \mapsto \lambda(x, y)$$

ker $\sigma = \{ z = \psi(x, y) \} \subseteq \bigcup_\lambda \Pi_\lambda$ imamo
 $\psi(x, y) = a(\lambda(x, y))x + b(\lambda(x, y))y = \psi(x, y, \lambda(x, y))$

normala za σ je paralelna $\vec{n}(\lambda)$, torej
 parametrizacija σ je $u(x, y) = (x, y, \varphi(x, y))$
 normala je podana z $u_x \times u_y$
 računamo:

$$u_x = (1, 0, \varphi_x(x, y)) = \varphi_x(x, y, \lambda(x, y)) + \varphi_\lambda(x, y, \lambda(x, y))\lambda_x(x, y)$$

$$u_y = (0, 1, \varphi_y(x, y)) = \varphi_y(x, y, \lambda(x, y)) + \varphi_\lambda(x, y, \lambda(x, y))\lambda_y(x, y)$$

$$-u_x \times u_y = (\varphi_x + \varphi_\lambda \lambda_x, \varphi_y + \varphi_\lambda \lambda_y, -1)$$

$$= (\varphi_x, \varphi_y, -1) + \varphi_\lambda (\lambda_x, \lambda_y, 0) =$$

$$\text{Spomnimo se } \psi = a(x)x + b(y)y$$

$$= \underbrace{(a(\lambda(x, y)), b(\lambda(x, y)), -1)}_{\vec{n}(\lambda(x, y))} + \varphi_\lambda'(\lambda_x, \lambda_y, 0)$$

Želimo dalje $\leftarrow = 0$

1. $(\lambda_x, \lambda_y, 0) = 0 \Rightarrow \nabla \lambda = 0 \Rightarrow \lambda$ konstantna
 $\Rightarrow \psi(x, y) = a(\lambda)x + b(\lambda)y \Rightarrow \sigma \subseteq \Pi_\lambda$ trivialna
 rešitev

2. $\varphi_\lambda(x, y, \lambda(x, y)) = 0 \quad \forall x, y \in P(\sigma)$

vseeno mi je, kajle } pri Hilbert

NELINEARNE ENAČBE PRVEGA REDA

$$F: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{V}$$
$$(x, z, p) \mapsto F(x, z, p)$$

$$\text{Enačbe: } F(x, z, \nabla u) = 0$$

Sistemi:

$$\dot{x} = \nabla_p F(x, z, p) = \left(\frac{\partial}{\partial p_1} F, \dots, \frac{\partial}{\partial p_n} F \right)$$

$$\dot{z} = p \cdot \nabla_p F(x, z, p)$$

$$\dot{p} = -(\nabla_x + p \nabla_z) F(x, z, p)$$

a)

$$(u_x u_y)^2 = u \quad u(x, 0) = x$$

$$(p_2)^2 = z$$

$$F(x, y, z, p, q) = (p_2)^2 - z$$

$$\dot{x} = \frac{1}{p_2} = 2(p_2)q$$

$$\dot{y} = 2(p_2)p$$

$$\dot{z} = 2p^2 q^2 + 2p^2 q^2 = 4p^2 q^2$$

$$\dot{p} = -(-p) = p$$

$$\dot{q} = q$$

$$p = C(s) e^t$$

$$q = D e^t$$

$$z = 4 \dot{z} \Rightarrow z = E(s) e^{4t}$$

$$\dot{z} = 4 e^{4t} C D$$

$$z = e^{4t} (D^2) = (e^{4t} - 1) C(s)^2 D(s)^2 + E(s)$$

$$\dot{x} = 2 C^2 D e^{3t} \Rightarrow x = \frac{2}{3} C^2 D (e^{3t} - 1) + F(s)$$

$$\dot{y} = \frac{2}{3} C D^2 e^{3t} \Rightarrow y = \frac{2}{3} C D^2 (e^{3t} - 1) + G(s)$$

$$u(x, 0) = x$$

$$X(s, 0) = s = F(s)$$

$$Y(s, 0) = 0 = G(s)$$

$$Z(s, 0) = 0 = E(s)$$

$$p(s, 0) = u_x(X(s, 0), Y(s, 0)) = 1 \Rightarrow C(s) = 1$$

$$q(s, 0) = \pm \sqrt{s}$$

$$(u_x u_y)^2(x, 0) = u(x, 0) = s$$

$$(1 \cdot u_y(x, 0))^2 = x$$

$$q_0 = u_y(x, 0) = \pm \sqrt{x}$$

$$u(x, 0) = \pm \sqrt{x} y + C(x)$$

$$z = (e^{4t} - 1)s + s$$

$$x = \frac{2}{3}(e^{3t} - 1)s + s \quad s = \frac{x}{1 + \frac{2}{3}(e^{3t} - 1)}$$

$$y = \frac{2}{3}(e^{3t} - 1)\sqrt{s}$$

$$p = e^t \quad e^{3t} = \frac{3}{2} \frac{y}{\sqrt{s}} + 1$$

$$q = e^t \sqrt{s}$$

$$s = \frac{x}{1 + \frac{y}{\sqrt{s}}}$$

$$s + \sqrt{s} y = x$$

$$s = \left(\frac{y}{\frac{2}{3}(e^{3t} - 1)} \right)^2$$

$$x = \frac{y^2}{\frac{2}{3}(e^{3t} - 1)} + \left(\frac{y}{\frac{2}{3}(e^{3t} - 1)} \right)^2$$

$$s = \left(\frac{x^2}{1 \pm \sqrt{y^2 - 4x}} \right)^2$$

$$v^2 x - v y^2 - y^2 = 0$$

$$D = \sqrt{y^4 - 4xy^2} = y \sqrt{y^2 - 4x}$$

$$\frac{2}{3}(e^{3t} - 1) = \frac{y^2 \pm y \sqrt{y^2 - 4x}}{x^2}$$

$$u = \frac{3}{2} \left(\frac{y^2 \pm y \sqrt{y^2 - 4x}}{x^2} \right) \left(\frac{x^2}{1 \pm \sqrt{y^2 - 4x}} \right)^2 + \left(\frac{x^2}{1 \pm \sqrt{y^2 - 4x}} \right)^2$$

$$b) u_x^2 + u_y = yu$$

$$u(x,0) = x$$

$$F(x,y,z,p,q) = p^2 + q - yz$$

$$\dot{x} = 2p$$

$$x(s,0) = s$$

$$\dot{y} = 1$$

$$y(s,0) = 0$$

$$\dot{z} = p \cdot (2p) + q \cdot 1$$

$$z(s,0) = s$$

$$\dot{p} = -0 + py = py$$

$$p(s,0) = 1$$

$$\dot{q} = z + qy$$

$$q(s,0) = -1$$

$$y = t + c(s) = t$$

$$dp = p t dt \leadsto \ln p = \frac{1}{2} t^2 dt + c \leadsto p = p_0 e^{\frac{1}{2} t^2} = e^{\frac{1}{2} t^2}$$

$$dx = 2e^{\frac{1}{2} t^2} dt \leadsto x =$$

$$2 \int e^{\frac{1}{2} t^2} dt = \int \sqrt{2} u^{-\frac{1}{2}} e^u du$$

$$u = \frac{1}{2} t^2$$

$$du = t dt \leadsto \sqrt{u} dt$$

$$\dot{z} = 2e^{t^2} + q$$

$$\dot{q} = z + q t$$

$$\ddot{z} = \ddot{z} + q + t \dot{q} = 2e^{t^2} + q$$

$$\ddot{q} + t \dot{q} = 2e^{t^2}$$

$$\text{hom: } \frac{d\dot{q}}{\dot{q}} = -t dt \leadsto \dot{q} = A(s) e^{-t}$$

$$\text{part: } A' e^{-t} = 2e^{t^2}$$

$$A' = 2e^{t^2+t}$$

$$b) u_x^2 + u_y = yu$$

$$u(x,0) = x$$

$$F(x,y,z,p,q) = p^2 + q - yz$$

$$\dot{x} = 2p$$

$$x(s,0) = s$$

$$\dot{y} = 1$$

$$y(s,0) = 0$$

$$\dot{z} = p \cdot (2p) + q \cdot 1$$

$$z(s,0) = s$$

$$\dot{p} = -0 + py = py$$

$$p(s,0) = 1$$

$$\dot{q} = z + qy$$

$$q(s,0) = -1$$

Linearne PDE 2. reda v $\mathbb{R}^2$

$$\underline{a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u + g(x,y) = 0}$$

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} u$$

$$\Delta = b^2 - ac = -\det \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

kanonična oblika

$\Delta > 0$ hiperbolični; tip

oziramo $u_{xy} + \text{nižji red} = 0$
 $u_{xx} - u_{yy} + \text{nižji red} = 0$

$\Delta = 0$ parabolični; tip

$$u_{xx} + \text{nižji red} = 0$$

$\Delta < 0$ eliptični; tip

$$u_{xx} + u_{yy} + \text{nižji red} = 0$$

Hiperbolični; tip:

$$\frac{\partial y}{\partial x} = \frac{b \pm \sqrt{\Delta}}{a} \leadsto y = f_{\pm}(x, c)$$

Novi koordinati $\xi(x, y), \eta(x, y)$

velja $y = f_+(x, \xi(x, y)), y = f_-(x, \eta(x, y))$

Primer: $\frac{\partial y}{\partial x} = 2 \pm 1 \leadsto y = \begin{cases} 3x + c = f_+ \\ x + c = f_- \end{cases}$

$$\xi(x, y) = y - 3x$$

$$\eta(x, y) = y - x$$

• če $a=0 \leadsto \frac{\partial x}{\partial y} = \frac{b \pm \sqrt{\Delta}}{c}$

• če $a=0$ in $c=0 \leadsto$ enačba je že v kanonični obliki

Parabolični tip $\Delta = 0$

$$\frac{dy}{dx} = \frac{b}{a} \quad (= \frac{b \pm \sqrt{\Delta}}{a}) \leadsto y = f(x, c)$$

$$y = f(x, \xi(x, y))$$

$\eta(x, y)$ je poljubna (ampak še koordinati)

Eliptični: tip $\Delta < 0$

$$\frac{dy}{dx} = \frac{b \pm i\sqrt{-\Delta}}{a} = \frac{b \pm \sqrt{\Delta}}{a} \leadsto y = f(x, c)$$

Iščemo $\Phi(x, y): y = f(x, \Phi(x, y))$

$$z = \operatorname{Re} \Phi \quad \eta = \operatorname{Im} \Phi$$

$$u_{xx} + 2u_{xz} + u_{yy} + u_{zz} = 0$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Diagonalisierung

$$(\lambda I - A) = \begin{bmatrix} \lambda - 1 & 0 & -1 \\ 0 & \lambda - 1 & 0 \\ -1 & 0 & \lambda - 1 \end{bmatrix} =$$

$$= (\lambda - 1)((\lambda - 1)^2 - 1) = (\lambda - 1)(\lambda^2 - 2\lambda) = \lambda(\lambda - 1)(\lambda - 2)$$

$$\lambda = 0: e_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\lambda = 1: e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda = 2: e_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 2 \end{bmatrix}$$

$$A = Q^T D Q$$

$$\xi_1 = x - z$$

$$\xi_2 = y$$

$$\xi_3 = x + z$$

$$\xi = Q \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$u = u(x(\xi), y(\xi), z(\xi)) = v(\xi)$$

ξ

$$x = \frac{1}{2}(\xi_1 + \xi_3)$$

$$u_x = (\xi_1)_x v_{\xi_1} + (\xi_2)_x v_{\xi_2} + (\xi_3)_x v_{\xi_3} \quad y = \xi_2$$

$$z = \frac{1}{2}(\xi_3 - \xi_1)$$

...

$$u_x = v_{\xi_1} + v_{\xi_3}$$

$$\partial_x = \partial_{\xi_1} + \partial_{\xi_3}$$

$$u_y = v_{\xi_2}$$

$$\partial_y = \partial_{\xi_2}$$

$$u_z = -v_{\xi_1} + v_{\xi_3}$$

$$\partial_z = \partial_{\xi_3} - \partial_{\xi_1}$$

$$u_{xx} = v_{\xi_1} + 2v_{\xi_1\xi_3} + v_{\xi_3}$$

$$u_{yy} = v_{\xi_2\xi_2}$$

$$u_{zz} = v_{\xi_1} - 2v_{\xi_1\xi_3} + v_{\xi_3}$$

$$u_{xz} = -v_{\xi_1\xi_1} + \dots$$

$$u_{xx} + 2u_{xz} + u_{yy} + u_{zz} = 0$$

$\Rightarrow$

$$4v_{\xi_2\xi_2} + v_{\xi_1\xi_1} = 0$$

ni kanonische
abf.hg

Moral: bi normierte Q

$$\tilde{\xi}_2 = \frac{\xi_2}{2}$$

$$\tilde{v}_{22} = \frac{\xi_2}{2}$$

$$v(\xi_1, \xi_2, \xi_3)$$

$$= v(\xi_1, \frac{\xi_2}{2}, \xi_3)$$

$$\tilde{v}_2 = v_2 \cdot \frac{1}{2}$$

$$\tilde{v}_{22} = v_{22} \cdot \frac{1}{4}$$

$$\Rightarrow \tilde{v}_{22} + v_{11} = 0$$

D'Alembertova formula na $(0, \infty) \times \mathbb{R}$

$$u(t, x) = \frac{1}{2}(f(x-t) + f(x+t)) + \frac{1}{2} \int_{x-t}^{x+t} g(z) dz + \frac{1}{2} \iint_{\Delta} F(\tau, z) d\tau dz$$

gde je $\Delta(t, x) = \{(\tau, z) \in \mathbb{R}^2 \mid 0 \leq \tau \leq t, |x-z| \leq t-\tau\}$

$$u_{tt} - u_{xx} = F(t, x)$$

$$u(0, x) = f(x)$$

$$u_t(0, x) = g(x)$$

$$u_{tt} - u_{xx} = 0 \quad \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

$$u(0, x) = u_t(0, x) = 0$$

$$k, \omega > 0$$

$$u_{tt} - u_{xx} = \sin(kx) \cos(\omega t)$$

$$u(0, x) = u_t(0, x) = 0$$

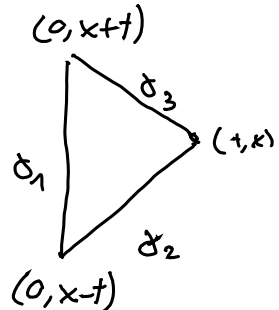
$$f(x) = g(x) = 0$$

$$F(x, t) = \sin(kx) \cos(\omega t)$$

$$u(t, x) = \frac{1}{2} \iint_{\Delta} F(\tau, z) d\tau dz$$

$$\Delta = \{(\tau, z) \in \mathbb{R}^2 \mid 0 \leq \tau \leq t, |x - z| \leq t - \tau\}$$

$$-\iint_D (y_x - x_y) dx dy = \int_{\partial D} (x, y) ds$$



$$X = -\frac{1}{\omega} \cos(kx) \cos(\omega t)$$

$$Y = 0$$

$$X_y = \sin(kx) \cos(\omega t)$$

$$\iint_{\Delta} \dots = - \iint_{\Delta} \left(-\frac{1}{\omega} \cos(kx) \cos(\omega t), 0 \right) ds$$

$$\delta_1 : 0$$

$$\delta_2 : \delta_2(s) = (s, x - (t-s)) \quad s \in (0, t)$$

$$\delta_3 = (t-s, x+s) \quad s \in (0, x)$$

$$\int_0^1 \left(\frac{1}{k} \cos(\omega s) \cos(k(x-t+s)), 0 \right) \cdot (1, 1) ds =$$

$$= \int_0^1 \frac{1}{2} (\cos(\alpha + \beta) + \cos(\alpha - \beta)) ds =$$

$$= \int_0^1 \frac{1}{2} (\cos(s(\omega + k) + x - t) + \cos(s(\omega - k) - x + t)) ds$$

$$u_{tt} - u_{xx} = 0 \quad u(0, x) = u_x(0, x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

d'Alembert's formula

ze verliest

$$u(x, y) = 0 + \frac{1}{2} \int_{-1}^1 1 dz + 0 = 1$$

$$\lim_{t \rightarrow \infty} = 1$$

$$u(x, t) = \frac{f(x+t) + f(x-t)}{2}$$

$$\sup = ?$$

$$u(x, t) \leq \frac{1}{2}(1+1) + \frac{1}{2} \cdot 2 + 0 = 2$$

$$x+t = m < 1 \quad x > 0 \Rightarrow t < 1$$

$$u(x, t) = \frac{1}{2}(1+1) + \frac{1}{2} \int_{x-t}^{x+t} g(z) dz$$

$$x=0 \quad t=m < 1$$

$$u(x, t) = 1 + \frac{1}{2} \int_{-m}^m g(z) dz = 1 + m$$

$$\lim_{m \rightarrow 1} u(x, t) = 1 \Rightarrow \sup = 2$$

$$u_{tt} - u_{xx} = 0$$

$$g(x) = 0 \quad t, x \in [0, \infty]$$

$$u(t, 0) = h(t)$$

$$u(t, 0) = h(t)$$

$$u(x, t) = \frac{f(x-t) + f(x+t)}{2}$$

$$u(t, 0) = \frac{1}{2} (f(t) + f(-t)) = h(t)$$

??

$$u_{tt} - u_{xx} = 0 \quad \begin{aligned} u(0, x) &= f(x) \\ u_t(0, x) &= 0 \\ u(t, 0) &= h(t) \end{aligned}$$

$$\delta = 0 + 1 = 1$$

$$\frac{dx}{dt} = \frac{0 \pm \sqrt{1}}{1} = \pm 1$$

$$x = \pm t + c$$

$$\begin{aligned} \zeta(t, x) &= x + t \\ \eta(t, x) &= x - t \end{aligned}$$

$$w_{\zeta\eta} = 0$$

$$w_{\zeta} = c(\zeta)$$

$$w = c(\zeta) + d(\eta)$$

$$u = \underbrace{c(x+t)}_F + \underbrace{d(x-t)}_G$$

$$x, t > 0$$

$$u(0, x) = F(x) + G(x) = f(x)$$

$$u(t, 0) = F(t) + G(-t) = h(t)$$

$$u_t(0, x) = F'(x) - G'(x) = 0$$

$$F'(x) = G'(x) \Rightarrow F(x) = G(x) + C$$

$$G(x) = \frac{f(x) - C}{2}$$

$$F(x) = \frac{f(x) + C}{2}$$

$$F(x) + G(x) = \frac{2f(x)}{2} \Rightarrow C = 0 \quad \text{per } x > 0$$

$$x < 0 \quad x > 0 \quad \text{vemo}$$

$$F(x) = \frac{1}{2} f(x)$$

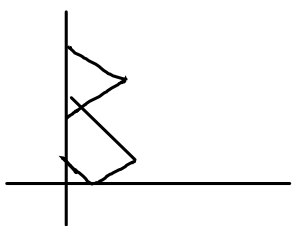
$$G(x) = \frac{1}{2} f(x)$$

$$F(x) = h(x) - G(-x) = \frac{1}{2} f(x)$$

$$\Rightarrow G(x) = h(-x) - \frac{1}{2} f(-x) \quad \text{per } x < 0$$

Splorna rezultat:

$$u(x, y) = \begin{cases} \frac{1}{2}(f(x+t) + f(x-t)) & ; \quad x-t > 0 \\ \frac{1}{2}f(x+t) - \frac{1}{2}f(-x+t) + h(-x+t) & ; \quad x-t < 0 \end{cases}$$



$$u_{tt} - u_{xx} = 0 \quad t > 0 \quad x \in \mathbb{R}$$

$$u(0, x) = x$$

$$u_t(0, x) = 0$$

isôlmo w , de bo $w_{tt} - w_{xx} = 0$

$$w = u + \frac{1}{12}x^4 - x$$

$$w_t = u_t$$

$$w_x = u_x + \frac{1}{3}x^3 - 1$$

$$w_{tt} = u_{tt}$$

$$w_{xx} = u_{xx} + x^2$$

$$w(0, x) = u(0, x) + \frac{1}{2}x^4 - x = \frac{1}{2}x^4$$

$$w_t(0, x) = u_t(0, x) = 0$$

$$w = F(x+t) + G(x-t)$$

$$w(x, 0) = F(x) + G(x) = \frac{1}{2}x^4$$

$$w_t(0, x) = F'(x) - G'(x) = 0$$

$$\leadsto F(x) = G(x) + C$$

$$2G(x) + C = \frac{1}{2}x^4$$

$$G(x) = \frac{1}{4}x^4 - \frac{1}{2}C$$

$$F(x) = \frac{1}{4}x^4 + \frac{1}{2}C$$

$$w(x, t) = \frac{1}{4}((x+t)^4 + (x-t)^4)$$

$$u(x, t) = \frac{1}{4}((x+t)^4 + (x-t)^4) - \frac{1}{2}x^4 + x$$

$$u, v \text{ smooth}; \quad u_{tt} - u_{xx} = 0 \quad t > 0$$

$$u_x(t, a) = u_x(t, b) = 0$$

$$u_t v_t + u_x v_x \stackrel{?}{=} F(x+t) + G(x-t)$$

$$u = f(x+t) + g(x-t)$$

$$v = h(x+t) + k(x-t)$$

$$\begin{aligned} u_t v_t + u_x v_x &= (f'(x+t) - g'(x-t))(h'(x+t) - k'(x-t)) \\ &\quad + (f'(x+t) + g'(x-t))(h'(x+t) + k'(x-t)) = \\ &= \underbrace{2f'(x+t)h'(x+t)}_{F(x+t)} + \underbrace{2g'(x-t)k'(x-t)}_{G(x-t)} \end{aligned}$$

$$F \stackrel{?}{=} t \mapsto \int_a^b u_t v_t + u_x v_x$$

$$\begin{aligned} F &\stackrel{?}{=} \int_a^b \frac{\partial}{\partial t} (u_{tt} v_t + u_t v_{tt} + u_{tx} v_x + u_x v_{tx}) dx = \\ &= \int_a^b (u_{tt} v_t + u_t v_{tt}) dx + \underbrace{u_t v_x \Big|_a^b}_0 - \int_a^b (u_t v_{xx} + u_{tx} v_t) dx \end{aligned}$$

$$= \int_a^b v_t (\underbrace{u_{tt} - u_{xx}}_0) + u_t (\underbrace{v_{tt} - v_{xx}}_0) dx = 0$$

$$u_x^2 u_{xx} \pm 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$$

$$\underbrace{\left(\frac{1}{3} u_x^3\right)_x}_{\left(\frac{1}{3} u_x^3\right)_x} \quad \underbrace{\left(\frac{1}{3} u_y^3\right)_y}_{\left(\frac{1}{3} u_y^3\right)_y}$$

$$u = f(x)g(y)$$

$$u_x = f'(x)g(y)$$

$$u_y = f(x)g'(y)$$

$$u_{xx} = f''(x)g(y)$$

$$u_{yy} = f(x)g''(y) \quad u_{xy} = f'(x)g'(y)$$

$$\Rightarrow f'(x)^2 g^3(y) f''(x) \pm 2f'(x)g(y)f(x)g'^2(y) + g'^2(y)f^3(x)g''(y)$$

$$\text{delimoz } f^3 g^3$$

$\Rightarrow$

$$\frac{f' f''}{f^3} \pm \frac{2f'(x)g'^2(y)}{g^2(y)f^2(x)} + \frac{g'^2 g''(y)}{g^3(y)} =$$

$$\text{Nehmen wir } f(x) = e^{\lambda x} \quad \Rightarrow \lambda e^{\lambda x} \Rightarrow \lambda^2 e^{\lambda x}$$

$$g(y) = e^{\mu y}$$

$$\lambda^4 \pm 2\lambda^2 \mu^2 + \mu^4 = 0$$

$$\Rightarrow (\lambda^2 \pm \mu^2)^2 = 0$$

$$\therefore (\lambda - \mu)(\lambda + \mu)(\lambda^2 + \mu^2) = 0 \quad \Rightarrow \lambda = \pm \mu$$

$$+ : \lambda^2 + \mu^2 = 0 \quad \Rightarrow \lambda = 0 \wedge \mu = 0$$

$$u) \quad u = e^{\lambda x} e^{\pm \lambda y} = e^{\lambda(x \pm y)}$$

$$\begin{aligned}
 u_{tt} - u_{xx} &= 0 & t \in (0, \infty) \\
 u(0, x) &= f(x) & x \in (0, 1) \\
 u_t(0, x) &= 0 \\
 u(t, 0) &= u(t, 1) = 0
 \end{aligned}$$

$$u(x, t) = F(x-t) + G(x+t)$$

$$\begin{aligned}
 u(0, x) &= F(x) + G(x) = f(x) \\
 u_t(0, x) &= -F'(x) + G'(x) = 0 \\
 u(t, 0) &= F(-t) + G(t) = 0 \\
 u(t, 1) &= F(1-t) + G(1+t) = 0
 \end{aligned}$$

$$\begin{aligned}
 F(x) &= G(x) + C \\
 F(x) &= \frac{f(x) - C}{2} & G(x) &= \frac{f(x) + C}{2} \quad \leadsto C=0 \leadsto \begin{aligned} F(x) &= \frac{f(x)}{2} \\ G(x) &= \frac{f(x)}{2} \end{aligned} \\
 \text{To je za } x \in (0, 1)
 \end{aligned}$$

$$-F(-t) = G(t) \leadsto f(-t) = -f(t) \quad \text{za } t \in (0, 1)$$

$$\begin{aligned}
 u(t, 1) &= F(1-t) = -G(1+t) \\
 t \in (0, 1) &\leadsto \\
 f(1-t) &= -G(1+t)
 \end{aligned}$$

$$G(t) = f(2-t) \quad \text{za } t \in (1, 2) \quad \begin{matrix} x=1+t \leadsto -t=1-x \\ \text{naa} \end{matrix}$$

$$t \in (n, n+1) \leadsto x \in (n, n+1)$$

$$G(1+t) = -F(1-t)$$

$$G(x) = -F(2-x)$$

$$G(x) = \begin{cases} -F(2-x) & ; x \geq 1 \\ \frac{1}{2}f(x) & ; x \in (0, 1) \\ -\frac{1}{2}f(-x) & ; x < 0 \end{cases}$$

$$t \in (0, 2) \leadsto F(1-t) = -G(1+t)$$

$$s \in (1, 3) \leadsto F(s) = -G(2-s) = -f(2-s) \cdot \frac{1}{2}$$

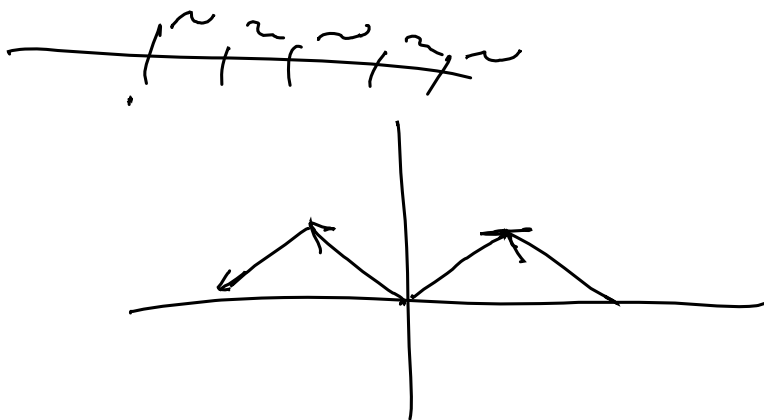
$$F(x) = \begin{cases} -\frac{1}{2}f(-x) & ; x \in (-1, 0) \\ \frac{1}{2}f(x) & ; x \in (0, 1) \\ -\frac{1}{2}f(2-x) & ; x \in (1, 2) \\ \frac{1}{2}f(2+x) & ; x \in (2, 3) \end{cases}$$

$$s = 1-t$$

$$t \in (-n-2, -n) \quad F(1-t) = -G(1+t)$$

$$\leadsto s \in (n-1, n+1) \quad \underline{F(s) = -G(2-s) = F(s-2)}$$

$$\begin{aligned}
 s \in (2n, 2n+1) &\leadsto F(s) = F(s-2) = F(s-2n) = \frac{1}{2}f(s-2n) \\
 s \in (2n-1, 2n) &\leadsto F(s) = F(s-2) = F(s-2n) = -\frac{1}{2}f(2n-s)
 \end{aligned}$$



$$u_t - ku_{xx} = 0 \quad \begin{array}{l} t \in (0, \infty) \\ x \in [0, 2\pi) \end{array}$$

$$u(t, 0) = u(t, 2\pi)$$

$$u_x(t, 0) = u_x(t, 2\pi)$$

$$u(0, x) = f(x)$$

$$u(x, t) = F(x - \sqrt{k}t) + G(x + \sqrt{k}t) \quad c = \sqrt{k}$$

$$u(0, x) = F(x) + G(x) = f(x)$$

$$u_x = F' + G'$$

$$F'(-ct) + G'(ct) = F'(2\pi - ct) + G'(2\pi + ct)$$

$$-F(-ct) + G(ct) = -F(2\pi - ct) + G(2\pi + ct) + D$$

$$u_t - u_{xx} = 0 \quad (t, x) \in (0, \infty) \times I \quad I \subset \mathbb{R}$$

z robnimi in začetnimi pogoji

Pišemo: $u(t, x) = v(t) w(x)$

$$\Rightarrow v'(t) w(x) = v(t) w''(x)$$

$$\Rightarrow \frac{v'}{v}(t) = \frac{w''}{w}(x) \equiv \lambda \in \mathbb{R}$$

$$\Rightarrow v(t) = c e^{\lambda t}$$

$$w(x) = \alpha e^{\sqrt{\lambda} x} + \beta e^{-\sqrt{\lambda} x} \quad \alpha, \beta \in \mathbb{C}$$

enačba je linearna $\Rightarrow$

$$u(t, x) = \sum_{k=0}^{\infty} v_k(t) w_k(x)$$

Odločimo $w_k(x) = \alpha_k e^{\sqrt{\lambda_k} x} + \beta_k e^{-\sqrt{\lambda_k} x}$, $v_k = c_k e^{\lambda_k t}$

Določimo λ_k odvisno od robnega pogoja
in c_k odvisno od začetnega pogoja

$$u(t, a) = u(t, b) = 0$$

$$V = \left\{ w \in L^2([a, b], \mathbb{R}); w(a) = w(b) = 0 \right\}$$

$$u_t - k u_{xx} = 0 \quad \begin{array}{l} t \in (0, \infty) \\ x \in [0, 2\pi] \end{array}$$

$$u(t, 0) = u(t, 2\pi)$$

$$u_x(t, 0) = u_x(t, 2\pi)$$

$$u(0, x) = f(x) = \sin x$$

$$u(t, x) = \sum v_k(t) w_k(x)$$

$$\begin{aligned} u(t, 0) = u(t, 2\pi) &\Rightarrow \sum v_k(t) w_k(0) = \sum v_k(t) w_k(2\pi) \\ &\Rightarrow \sum v_k(t) (w_k(0) - w_k(2\pi)) = 0 \end{aligned}$$

$$\begin{aligned} u_x(t, x) = \sum v_k(t) w'_k(x) &\Rightarrow \\ \sum v_k(t) (w'_k(0) - w'_k(2\pi)) &= 0 \end{aligned}$$

$$u(0, x) = f(x) \Rightarrow \sum v_k(0) w_k(x) = f(x)$$

$$v_k = c_k e^{\lambda_k t} \leadsto \sum c_k e^{\lambda_k t} (w_k(0) - w_k(2\pi)) = 0$$

$$w_k(0) = \alpha_k + \beta_k = 0$$

$$w_k(\pi) = \alpha_k e^{\sqrt{\lambda_k} \pi} + \beta_k e^{-\sqrt{\lambda_k} \pi} = 0$$

$$e^{\sqrt{\lambda_k} \pi} - e^{-\sqrt{\lambda_k} \pi} = 0$$

$$e^{2\sqrt{\lambda_k} \pi} = 1 \Rightarrow 2\sqrt{\lambda_k} \pi = 2\pi i k$$

$$\sqrt{\lambda_k} = i k$$

$$\lambda_k = -k^2$$

Parametrove samo razneje med α in β

$$w_k = e^{ikx} - e^{-ikx} \leadsto \text{BSES} \leadsto w_k = 2i \sin kx$$

$$\underline{\underline{w_k = \sin(kx)}}$$

$$u(t, x) = \sum_{k=1}^{\infty} c_k e^{-k^2 t} \sin(kx)$$

$$u(0, x) = \sum_{k=1}^{\infty} c_k \sin(kx) = \sin x \xrightarrow{\text{keys}} c_k = 0 \text{ za } k \neq 1$$

$$u(t, x) = e^{-t} \sin x$$

Mifhom de sons
menhele

①

$$u_{tt} - c^2 u_{xx} = F(t, x)$$

$$u_x(t, 0) = t^2$$

$$u(t, L) = -t$$

$$u(0, x) = x^2 - L^2$$

$$u_t(0, x) = \sin^2\left(\frac{\pi x}{L}\right)$$

u_1, u_2 rešitvi:

$$v = u_1 - u_2$$

$$v \equiv 0$$

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= u_{1tt} - u_{2tt} - c^2 u_{1xx} + c^2 u_{2xx} = \\ &= F(t, x) - F(t, x) = 0 \end{aligned}$$

$$v_x(t, 0) = t^2 - t^2 = 0$$

$$v(t, L) = -t + t = 0$$

$$v(0, x) = 0$$

$$v_t(0, x) = 0$$

$$\varphi(t) = \int_0^L (v_t^2 + c^2 v_x^2) dx$$

$$\varphi'(t) = \int_0^L (2v_t v_{tt} + 2c^2 v_x v_{xt}) dx =$$

$$= 2 \int_0^L v_t v_{tt} dx + 2c^2 \int_0^L v_x v_{xt} dx$$

$$= 2 \int_0^L v_t v_{tt} dx + 2c^2 (v_x v_t - \int_0^L v_{xx} v_t dx) =$$

$$v_{tt} = c^2 v_{xx}$$

$$c^2 2 \int_0^L v_t v_{xx} + 2c^2 v_x v_t \Big|_0^L - 2c^2 \int_0^L v_{xx} v_t dx =$$

$$= 2c^2 (v_x(t, L) \cdot v_t(t, L) - v_x(t, 0) v_t(t, 0)) =$$

$$= 0$$

$$\Rightarrow \varphi(t) = \varphi(0) = \int_0^L (v_t^2(0, x) + c^2 v_x^2(0, x)) dx =$$

$$= \int_0^L (0 + c^2 \cdot 0) dx = 0$$

$$\Rightarrow v_t^2 + v_x^2 = 0 \Rightarrow v_x = 0 \quad v_t = 0$$

$$\Rightarrow v = C(y) \quad \wedge \quad v = C(x) \Rightarrow v = \overset{\uparrow}{\text{konstanta}}$$

$$\Rightarrow v = 0 \text{ zaradi robnih pogojev}$$

What is a solution to PDE

$$F(x, u, \nabla u, \dots) = 0$$

A solution would be a function u s.t.

$$F(x, u, \nabla u, \dots) = 0$$

Since the equation involves derivatives of u up to some order $n \in \mathbb{N}$, we usually assume that the derivatives exist and are continuous

Def: A **classical solution** on an open set $\Omega \subseteq \mathbb{R}^2$ is $u \in C^n(\Omega)$ s.t. $F(x, u, \nabla u, \dots) = 0$

Standard questions:
- existence
- uniqueness
- maximal Ω
} usually related to the regularity of F

V 1. wa

7.5

$$u, v \in C^{1,2}((0, T) \times (0, L)) \cap C([0, T] \times [0, L])$$

$$u_t - u_{xx} \leq v_t - v_{xx} \quad \text{na } (0, T) \times (0, L)$$

$$u \leq v \quad \text{na parabolickém robu} \\ \{0\} \times [0, L] \cup [0, T] \times \{0, L\}$$

$$\underline{u \leq v \quad \text{na } [0, T] \times [0, L] = \Omega_T} \quad \text{"\(\Gamma\)"}$$

Obraťme vzhľadom $u_t - v_t - u_{xx} + v_{xx} =$
 $= (u - v)_t - (u - v)_{xx} \leq 0$
 na robu: $u - v \leq 0$
 (parabolickom)

$$\Rightarrow \max_{\Omega_T \cup \Gamma_T} (u - v) = \max_{\Gamma_T} (u - v)$$

$$\max_{\Omega_T} (u - v) = \max_{\partial(\Omega_T)} (u - v) \leq 0$$

$$\Rightarrow u \leq v \quad \text{na } \Omega_T$$

Harmonične funkcije nima izoliranih ničel $n=2$

$$u_{xx} + u_{yy} = 0$$

$$u(x_0, y_0) = 0$$

$$u(x, y) = u(x + iy) = \frac{1}{2\pi} \int_{S_r} u(\alpha, \beta) dS_{\alpha, \beta}$$

če je izolirana ničla, obkroži dovolj majhno
disko, da nima ničel, torej u ne spreminja
predznaka

$$\frac{1}{2\pi r} \int_0^1 \underbrace{u(\gamma(t))}_{\neq 0} \underbrace{|\dot{\gamma}(t)|}_{\neq 0} dt \neq 0$$

$$\Delta u(x) = 0 \Rightarrow \Delta u(Mx+b) = 0, \forall b \in \mathbb{R}^n \text{ in } \dots \dots \dots \forall M \text{ orthogonal}$$

$$u(x) = \sum_{i=1}^n u_{x_i x_i}(x)$$

$$\Delta(u(Mx+b)) = \sum_{i=1}^n \delta_{x_i}^2 u(Mx+b) =$$

$$= \sum_{i=1}^n \delta_{x_i}^2 (u(\sum m_{jk} x_k + b_j)) =$$

$$= \sum \delta_{x_i} \left(\sum_j u_{x_j} ([\dots]_j) \cdot m_{ji} \right) =$$

$$= \sum_i \sum_j \sum_l u_{x_j x_l}(Mx+b) \cdot \underbrace{m_{ji} \cdot m_{li}}_{\substack{=0 \text{ } j \neq l \\ =1 \text{ } j=l}} =$$

$M \cdot M^T = I$

$$= \sum_j \sum_l u_{x_j x_l}(Mx+b) \delta_{jl} =$$

$$= \sum_i u_{x_i x_i}(Mx+b) = \underset{0}{(\Delta u)}(Mx+b) = 0$$

$$\textcircled{2} \quad P(x,y) = \sum_{j=0}^n a_j x^j y^{n-j}$$

$$V_n = \{ \sum_{j=0}^n a_j x^j y^{n-j} \mid a_j \in \mathbb{R} \} \quad \dim V_n = ?$$

$$\partial P(x,y) = \partial \sum a_j (j x^{j-1} y^{n-j} + (n-j) x^j y^{n-j-1}) =$$

$$= \sum a_j (j(j-1) x^{j-2} y^{n-j} + (n-j)j x^{j-1} y^{n-j-1} + j(n-j) x^{j-1} y^{n-j-1} + (n-j)(n-j-1) x^j y^{n-j-2}) =$$

$$= \sum_{j=2}^n a_j x^{j-2} y^{n-j-2} \underbrace{(j(j-1)x^2 + 2j(n-j)xy + (n-j)(n-j-1)x^2)}_{\neq 0}$$

$$V_0 \Rightarrow \text{konst} \Rightarrow \dim V = 1$$

$$V_1 \Rightarrow a_0 x + a_1 y \Rightarrow \dim V = 2$$

Psst, helfe em se zurechtfinden

Ponovimo

$$\Delta u = 0 \quad \text{ne } D$$

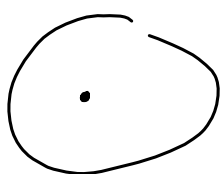
$$u = f \quad \text{ne } \partial D$$

$$P_r(t) = \frac{1 - r^2}{1 - 2r \cos(2\pi t) + r^2}$$

$$\Rightarrow u(r \cos(2\pi t), r \sin(2\pi t)) = (P_r * f)(t) = \int f(x) P_r(t-x) dx$$

$$③ \quad U \subseteq \mathbb{R}^2 \quad a \in U$$

$\forall$ omejeno herm funkc $u \in C^2(U - \{a\})$
 razširimo do harmonične na u



izberemo najpovprek $D \subset U$

Označimo: $f = u|_D$ razširimo f do harmonične
 na $U \supseteq U'$ $\Delta v = 0$ na ∂

$$u - v \equiv 0$$

$$(u - v)|_{\partial D} = 0$$

$$\Delta(u - v) = 0 \quad \text{če je } u - v \text{ zveza} \Rightarrow u \equiv v$$

Rešitev na $\partial \Delta(\overset{w}{u - v}) = 0$ na

Rešimo da v a ne moremo razširiti
 do uveznosti



izberemo $a \in \mathbb{R}^2$

to je izberemo ∂

na $D = K(a, r)$ je ∂ $\neq \emptyset$

W. d. zvezi uveznost na $K(a, r)$

ko p. g. p. p. a. m. b. b. i.
 uveznost na U a

$$\log |z - a| \quad \log \left| \frac{z - a}{r} \right|$$

$$W_\delta = W + \delta \cdot \log \left| \frac{z - a}{r} \right| \quad \delta \geq 0$$

ker je W omejena

$$\max_{\partial D} W_\delta \leq 0 \quad \text{za dovolj majhen } \delta$$

$$\Rightarrow W_\delta \leq 0 \quad \text{na } \partial D \text{ za dovolj majhen } \delta$$

$$W_\delta \leq 0 \quad \text{na } D - \{a\}$$

$$\text{p. g. p. } \delta \rightarrow 0 \Rightarrow W \leq 0 \quad \text{na } D - \{a\}$$

$$\text{če } \delta < 0 \text{ jo } W_\delta \geq 0$$

$$\Rightarrow \delta \rightarrow 0 \quad W \geq 0$$

$$\Rightarrow W = 0 \quad \text{na } D - \{a\} \Rightarrow u \text{ lahko}$$

razširimo na U

②

u zerna do broba $u, 0$

$$\frac{1-|p|}{1+|p|} u(0,0) \leq u(p) \leq \frac{1+|p|}{1-|p|} u(0,0)$$

za $\forall p \in \mathbb{D}$

razširimo do $\mathbb{D} = \mathbb{P}$

$$|p| \rightarrow 1 \quad |p| e^{2\pi i t} \leadsto e^{2\pi i t}$$

$$p: re^{2\pi i t} \mapsto \frac{r}{|p|} e^{2\pi i t}$$

$$u = v \circ f = F$$

τ od 0 do p

$$u(0,0) = v(0,0)$$

$$\frac{1-|p|}{1+|p|} v(0,0) \leq v(e^{2\pi i t}) \leq \frac{1+|p|}{1-|p|} v(0,0)$$

$$v(re^{2\pi i t}) = \mathcal{P}_r \circ u(t) \Big|_{|p|=1} = \int_0^1 u(pe^{2\pi i \tau}) \mathcal{P}_r(t-\tau) d\tau =$$

$$= \int_0^1 u(pe^{2\pi i \tau}) \frac{1-|r|^2}{1-2|r|\cos 2\pi(t-\tau)+|r|^2} d\tau$$

$$v(0,0) = \int_0^1 u(pe^{2\pi i \tau}) d\tau$$

$$u(re^{in\tau}) = \int_0^1 u(e^{2\pi i \tau}) \frac{1-|p|^2}{1-2|p|\cos(2\pi(t-\tau))+|p|^2} d\tau$$

$$\begin{aligned} \frac{1-|p|^2}{1-2|p|\cos(2\pi(t-\tau))+|p|^2} &\leq \frac{1-|p|^2}{1-2|p|+|p|^2} = \\ &= \frac{(1-|p|)(1+|p|)}{(1-|p|)^2} = \\ &= \frac{1+|p|}{1-|p|} \end{aligned}$$

$$\int_0^1 u(e^{2\pi i \tau}) d\tau \cdot C_1 = u(0,0) \cdot C_1$$

...

$\Rightarrow$ adaguar

$$\textcircled{5} \quad H = \mathbb{R} \times (0, \infty) \quad u \in \mathcal{C}(\overline{H}) \cap C^2(H)$$

$$\sup_{\overline{H}}(u) = \sup_{\partial H} u$$

$$\text{Naj bo } \Omega(r) = \{x, y \in H; x^2 + y^2 < r^2\}$$

Vemo: $u|_{\Omega}$ doseže maksimum na $\partial\Omega$

$$v = u + \gamma \varphi$$

Potrebujemo $\lim_{r \rightarrow \infty} \rightarrow -\infty$

$$\varphi = -\log(r) = -\log(x^2 + (y+1)^2)$$

isto kot prej

$$\Delta u = 0 \quad \text{in } \mathbb{H}$$

$$u = f \quad \text{on } \partial \mathbb{H}$$

$$\varphi: \mathbb{H} \longrightarrow \mathbb{D}$$

$$z \mapsto \frac{z-i}{z+i}$$

$$v(re^{i\theta}) = \int_0^1 \tilde{f}(e^{i\theta\tau}) P_r(t-\tau) d\tau = v(\varphi(z)) = u(z)$$

$$u(z) = v(\varphi(z)) =$$

$$P_r(t) = \frac{1-r^2}{1-2r\cos(2\pi t)+r^2}$$

$$u(x, 0) = 0$$

u harmonische

$$v(x, y) = \begin{cases} u(x, y) & y > 0 \\ -u(x, -y) & y \leq 0 \end{cases}$$

$$\Delta v = \begin{cases} u_{xx} + u_{yy} & y > 0 \\ -u_{xx} - u_{yy} & y \leq 0 \end{cases} = 0$$

Bzgl $\bar{z}$ in v wieder

Ab; ima v LPV $\geq$

prüfen: merkmale ~~(x, y)~~ $y=0$

$$\frac{1}{2\pi r} \int_0^1 u(z_0 + re^{2\pi i t}) dt = 0 \quad \text{za } \forall r \leq r_0 =$$

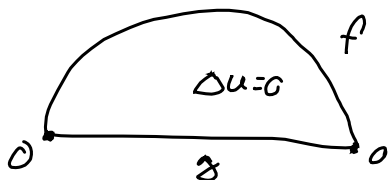
$$= \frac{1}{2\pi r} \left(\int_0^{\frac{1}{2}} u(z_0 + re^{2\pi i t}) dt - \int_{\frac{1}{2}}^1 u(z_0 + re^{-2\pi i t}) dt \right)$$

$$t = \tau + \frac{1}{2}$$

$$= \frac{1}{2\pi r} \left(\int_0^{\frac{1}{2}} dt - \int_0^{\frac{1}{2}} u(z_0 + re^{-2\pi i \tau} \cdot e^{-\pi i}) d\tau \right) =$$

$$= \frac{1}{2\pi r} \left(\int_0^{\frac{1}{2}} dt - \int_0^{\frac{1}{2}} u(z_0 + re^{2\pi i \tau}) d\tau \right) = 0 \quad \therefore$$

$$\begin{aligned} \partial u &= 0 \quad \text{na } B^+ \\ u &= f \quad \text{na } \partial B_1(0) \cap \end{aligned}$$



izdejati razširjeno pravilo 10 B

2D

$$v = \begin{cases} f & y > 0 \\ h & y < 0 \end{cases}$$

$$\begin{aligned} v(x, y) &= f * P_r = \int_0^1 f(t - \tau) \frac{1 - r^2}{1 - 2r \cos(2\pi(t - \tau)) + r^2} d\tau \\ &= \int_0^{\frac{1}{2}} f(\tau) \frac{1 - r^2}{1 - 2r \cos(2\pi(t - \tau)) + r^2} d\tau + \\ &\quad + \int_{\frac{1}{2}}^1 h(\tau) \frac{1 - r^2}{1 - 2r \cos(2\pi(1 - \tau)) + r^2} d\tau \end{aligned}$$

Vstavimo $y = 0$

$$\begin{aligned} &\int_0^{\frac{1}{2}} f(\tau) \frac{1 - r^2}{1 - 2r \cos \cos(2\pi\tau) + r^2} d\tau + \\ &+ \int_{\frac{1}{2}}^1 h(\tau) \frac{1 - r^2}{1 - 2r \cos(2\pi\tau) + r^2} d\tau = g(r) \end{aligned}$$

Rezultat za $\partial \tilde{D}$

$$\Delta V_2 = 0 \text{ na } \partial D$$

$$V_2 = \tilde{f} \text{ na } \partial D$$

Rezultat za $\underline{\hspace{2cm}}$

$$\Delta V_1 = 0$$

$$V_1 = \tilde{g} \text{ na } \partial H$$

Ideja rešiti za v_1 pr. in pr. v_1 na
početi za v_2

$$V_1 = \frac{y}{\pi} \cdot \int_{-\infty}^{\infty} \frac{1}{(x-s)^2 + y^2} \tilde{g}(s, 0) ds \dots \text{rep. reprezentativna formula}$$

$$V_1 = \frac{y}{\pi} \int_{-1}^1 \frac{1}{(x-s)^2 + y^2} \tilde{g}(s, 0) ds$$

$$\Delta V_2 = 0 \text{ na } \partial H$$

$$v_2 = \tilde{f} - v_1$$

$$v_2 = re^{i\theta} = (P_r * (\tilde{f} - v_1))(e^{i\theta})$$

$w = v_1 + v_2$ reši problem. samo preveriti
mešamo

$$v_2 = 0 \text{ na } \partial D \cap \{y=0\}$$

$$v_2(x, 0) = v_2(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} (\tilde{f} - v_1)(\cos \theta, \sin \theta) P_r(\theta) d\theta$$

$$P_r(t) = \frac{1-r^2}{1+r^2-2r\cos t}$$

$$\int_0^{2\pi} (\tilde{f} - v_1)(\cos(t), \sin(t)) P_r(t) dt$$

h

$u: \mathbb{R}^n \rightarrow \mathbb{R}$ harmonizē

$$\exists c \in \mathbb{R}_{>0}: \frac{|u(p)|}{\sqrt{1+|p|}} \leq c \quad \forall p$$

detasā: daļē u konstanta

$$\Delta u = 0 \quad \frac{\partial}{\partial x_i} \Delta u = \Delta \frac{\partial}{\partial x_i} u = 0$$

odvodi se harmonizē

$$F = (0, 0, \dots, \vec{e}_i, 0, \dots, 0)$$

v toču p:

$$\frac{\partial}{\partial x_i} u = \frac{1}{|B(r, p)|} \int_{B(r, p)} \frac{\partial}{\partial x_i} u(y) dy \leq \frac{1}{|B(r, p)|} \int \operatorname{div} F(y) dy$$

$$\frac{|S_r(p)|}{|B_r(p)|} = \frac{n}{r}$$

$$= \frac{1}{|B(r, p)|} \int_{S(r, p)} \vec{F} \cdot \vec{n} dy = \frac{1}{|B|} \int_S \vec{F} \cdot \frac{\vec{x} - p}{r} dS_{x_i}$$

$$|\frac{\partial}{\partial x_i} u| \leq \frac{1}{|B|} \int_S |u| \left| \frac{\vec{x} - p}{r} \right| dS_{x_i}$$

$$\leq \frac{1}{|B|} \int_S \sup_S |u| dS_{x_i} = \frac{n}{r} \sup_S |u|$$

$$\leq \frac{nc}{r} \sqrt{1+|p+r|}$$

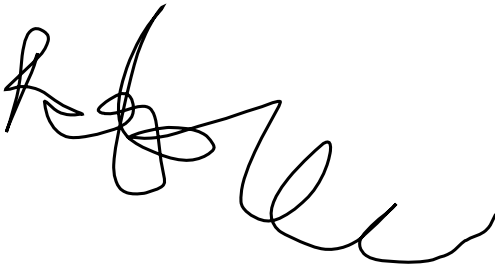
$$r \rightarrow \infty \quad \downarrow$$

$u \in L^1_{loc}(I)$. Potem $v \in L^1_{loc}(I)$ je šibki odvod
 u če $\int_I u(x) \varphi'(x) dx = - \int_I u(x) \varphi(x) dx \quad \forall \varphi \in C_c^\infty(I)$

$$L_{loc} = \{u: I \rightarrow \mathbb{R} : \forall p \in \mathbb{Z} \exists \varepsilon > 0 \int_{p-\varepsilon}^{p+\varepsilon} |u(x)| dx < \infty\}$$

Soboljev prostor

$$W^{k,p}(I) = \{u \in C^p(I) : \text{šibki } k\text{-ti odvod } u\}$$



$$I = (-1, 1)$$

$$u(x) = |x|$$

rozważmy sobie adwokat

$$\begin{aligned} \int_I u(x) \varphi'(x) dx &= \int_{-1}^1 |x| \varphi'(x) dx = \int_0^1 x \varphi'(x) dx - \int_{-1}^0 x \varphi'(x) dx \\ &= x \varphi(x) \Big|_0^1 - \int_0^1 \varphi(x) dx - x \varphi(x) \Big|_0^{-1} + \int_{-1}^0 \varphi(x) dx = \\ &= \underbrace{\varphi(1)}_0 + \underbrace{\varphi(-1)}_0 - \int_0^1 \varphi(x) dx + \int_{-1}^0 \varphi(x) dx \\ v(x) &= \begin{cases} 1 & ; x \in (0, 1) \\ -1 & ; x \in (-1, 0) \end{cases} \end{aligned}$$

$$I = (0, 2) \quad a(x) = \begin{cases} 1 & 0 < x < 1 \\ 2 & 1 < x < 2 \end{cases}$$

$$-(a(x)u'(x))' = 1 \quad \text{ne } I \quad u(0) = u(2) = 0$$

$$-a' u' - a u'' = 1$$

$$\int a(x) p'(x) dx = - \int a'(x) \varphi(x) dx$$

Isöemo sibe reģiter

$$-a(x)u'(x)p(x) \Big|_0^2 + \int_0^2 a(x)u'(x)p'(x) dx =$$

$$= -a(2)u'(2)p(2) + a(0)u'(0)p'(0) + \\ + \int_0^1 a(x)u'(x)p'(x) dx + \int_1^2 a(x)u'(x)p'(x) dx =$$

$$= -a(2)u'(2)p(2) + a(0)u'(0)p'(0) + \\ \int_0^1 u'(x)p'(x) + 2 \int_1^2 u'(x)p'(x)$$



~~ue~~

$\Rightarrow$

$$z(x) = \begin{cases} u'(x) ; \\ 2u'(x) ; \end{cases} = a(x)u'(x)$$

$$\int a(x)u'(x)p'(x) dx = - \int (\dots) p(x) dx$$

$$a(x)u'(x)$$

$$X_1, \dots, X_n \sim U(a, b)$$

a) $a, b = ?$ po metodi največjega verjetja

$$L(a, b | X_1, \dots, X_n) = \prod_{i=1}^n \mathbb{1}_{a < X_i < b} \frac{1}{b-a} =$$

$$= \mathbb{1}_{a \leq X_1, \dots, X_n \leq b} \prod_{i=1}^n \frac{1}{b-a}$$

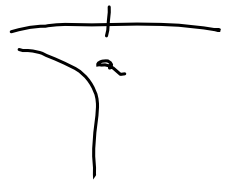
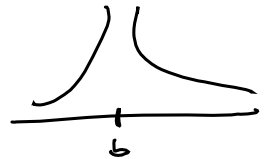
$$a < X_1:$$

$$\cancel{L(a, b | X_1, \dots, X_n)} = \cancel{\prod_{i=1}^n \frac{1}{b-a}}$$

$$l(a, b | X_1, \dots, X_n) = \sum_{i=1}^n \frac{1}{b-a}$$

$$\frac{\partial}{\partial a} l = \sum_{i=1}^n (b-a)^{-2} = 0 \quad \left(\frac{1}{(b-a)^2} \right)$$

$$\frac{\partial}{\partial a} l = \sum_{i=1}^n (b-a)^{-2} = 0 = \frac{1}{(b-a)^2}$$



L doseže maksimum ko sta b in a bližje

$$\Rightarrow \hat{a} = \min \{X_1, \dots, X_n\} \quad \hat{b} = \max \{X_1, \dots, X_n\}$$

$$\mathbb{E}(\hat{a} - a) \geq 0$$

$$\mathbb{P}(\hat{a} = a) = \mathbb{P}(X_1 = a) = 0$$

$$\Rightarrow \mathbb{E}(\hat{a} - a) > 0$$

$$\mathbb{E}(\hat{b} - b) < 0$$

$$X_i \sim U[\vartheta - c, \vartheta + c]$$

$$f(x|\vartheta) = \begin{cases} \frac{1}{2c} & ; \vartheta - c \leq x \leq \vartheta + c \\ 0 & ; \text{sonst} \end{cases}$$

$$L(\vartheta | x_1, \dots, x_n) = \prod_{i=1}^n \frac{1}{2c} \quad \leftarrow \begin{array}{l} \text{konst} \\ \text{heterokoll center,} \\ \text{da ja} \end{array}$$

$$\Rightarrow \vartheta - c \leq x_1, \dots, x_n \leq \vartheta + c$$

$$l(\vartheta | \dots) = \sum \frac{1}{2c} \neq 0$$

$$\vartheta - c \leq x_i \leq \vartheta + c$$

$$-c - x_i \leq -\vartheta \leq -x_i + c$$

$$|\vartheta - x_i| < c$$

$$\vartheta = \frac{x_1 - x_n}{2}$$

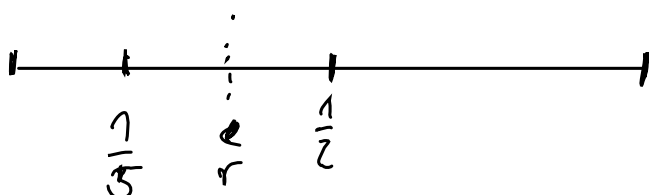
$$\begin{aligned} \mathbb{E}\left(\frac{x_1 - x_n}{2} - \vartheta\right) &= \mathbb{E}\left(\frac{x_1 - x_2}{2} - \frac{x_1 + \dots + x_n}{n}\right) \\ &= \mathbb{E}\left(\frac{(n-2)x_1 + (n-2)x_n}{2n} - x_2\right) \end{aligned}$$

$$\frac{1}{2} \mathbb{E}(\max\{x_1, \dots, x_n\} - \min\{x_1, \dots, x_n\}) = \vartheta$$

$$X \sim \text{Geom}(p)$$

$$H_0: p = \frac{1}{2} \quad \alpha = 0,05$$

$$H_1: p = \frac{1}{5}$$



$$H_0 \text{ zavrnemo, če je } \hat{p} < c \in \left[\frac{1}{5}, \frac{1}{2}\right]$$

$$\mathbb{P}(X=k) = p g^{k-1}$$

$$\mathbb{E}(X) = \frac{1}{p} \Rightarrow \hat{p} = \frac{1}{X} < r$$

$$\sum_{n=1}^{\infty} X_n > n r$$

$$X > c$$

$$\text{Opazimo } X=k$$

$$p g^{k-1} = r$$

$$\alpha \geq \mathbb{P}(X > c) = \sum_{k=c}^{\infty} p g^k = p g^c \sum_{k=0}^{\infty} g^k = p g^c \frac{1}{1-g} = g^c$$

$$\alpha \geq g^c \quad \alpha = g^c$$

$$X = \mathbb{E}_g(X > c) = p g^c = p \alpha$$

$$r = \frac{\alpha}{X} \text{ torej, če je } \hat{p} < \frac{\alpha}{X}$$

$$H_0: p = \frac{1}{2}$$

$$H_1: p = \frac{1}{5}$$

Stopnja tveganja:

$$P(H_0 \text{ zavrnemo} \mid H_0 \text{ drži}) \leq \alpha$$

Zavrnemo za $X \geq c$

$$\alpha \geq P(X \geq c \mid p = \frac{1}{2}) = \sum_{k=c}^{\infty} \binom{c-1}{k} \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^c \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{2^c} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2^{c-1}}$$

$$\alpha \geq \frac{1}{2^{c-1}} \Rightarrow 2^{c-1} \geq \frac{1}{\alpha}$$

$$c-1 \geq \log_2 \frac{1}{\alpha}$$

$$c \geq \log_2 \frac{1}{\alpha} + 1$$

stopnja tveganja v odvisnosti od c :

$$P(X \geq c \mid p = \frac{1}{2}) = \frac{1}{2^{c-1}}$$

za $c = \log_2 \frac{1}{\alpha} + 1$: Stopnja tveganja: α

Moč preizkusa:

$$P(H_0 \text{ zavrnemo} \mid H_1 \text{ drži})$$

$$P(X \geq c \mid p = \frac{1}{5}) = \sum_{k=c}^{\infty} \frac{1}{5} \left(\frac{4}{5}\right)^k = \frac{1}{5} \cdot \left(\frac{4}{5}\right)^{c-1} \cdot \frac{1}{1-\frac{4}{5}}$$

$$= \left(\frac{4}{5}\right)^{c-1}$$

Moč preizkusa je največja
za čim večji c

$$\text{za } c = \log_2 \frac{1}{\alpha} + 1: \left(\frac{4}{5}\right)^{\log_2 \frac{1}{\alpha}} + 1$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right) \mathbb{P}(x > 0 \vee y > 0)$$

$$\begin{array}{ccc} \mathbb{P}(x > 0) & + & \mathbb{P}(y > 0) - \mathbb{P}(x > 0 \wedge y \geq 0) \\ \parallel & & \parallel \\ \frac{1}{2} & & \frac{1}{2} \end{array}$$

$$\mathbb{P}(x > 0 \wedge y > 0) = \int_{\substack{x > 0 \\ y > 0}} f_{x,y}(x,y) dx dy$$

$$Y = X\beta + \varepsilon$$

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

$$Y = X \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \varepsilon_i$$

$$X = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \quad \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} & \end{bmatrix}$$

$$\beta = ? \quad \beta = (X^T X)^{-1} X^T Y$$

$$X^T X = \begin{bmatrix} 6 & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} = \begin{bmatrix} 6 & 24 \\ 24 & 106 \end{bmatrix}$$

$$\det(X^T X) = 60$$

$$X^T Y = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix} = \begin{bmatrix} 43 \\ 180 \end{bmatrix}$$

$$\beta = \frac{1}{60} \begin{bmatrix} 106 & -24 \\ -24 & 6 \end{bmatrix} \begin{bmatrix} 43 \\ 180 \end{bmatrix} = \frac{1}{60} \begin{bmatrix} 238 \\ 48 \end{bmatrix}$$

$$\frac{1}{60} \begin{bmatrix} 238 \\ 48 \end{bmatrix} = \begin{bmatrix} 3,97 \\ 0,8 \end{bmatrix}$$

$$\hat{\sigma}^2 = \frac{\|Y - X\beta\|^2}{6-}$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = A \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

$$\text{var} \begin{bmatrix} X \\ Y \end{bmatrix} = \cancel{A} A^T \text{var} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \cancel{A} A^T$$

$$\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ 0 & d \end{bmatrix} = \begin{bmatrix} a^2 & ac \\ ac & c^2 + d^2 \end{bmatrix}$$

$$a = 1$$

$$c = \rho \quad d = \sqrt{1 - \rho^2}$$

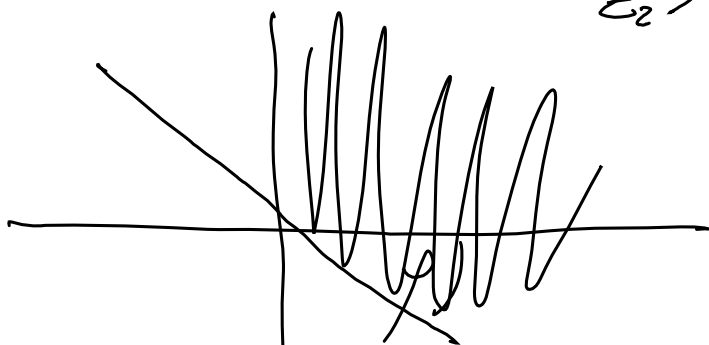
$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1 - \rho^2} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

$$X = Z_1$$

$$Y = \rho Z_1 + \sqrt{1 - \rho^2} Z_2$$

$$\mathbb{P}(X > 0 \wedge Y > 0) = \mathbb{P}(Z_1 > 0 \wedge \rho Z_1 > -\sqrt{1 - \rho^2} Z_2)$$

$$Z_2 > -\frac{\rho}{1 - \rho^2} Z_1$$



$$\frac{1}{4} - \frac{\arctan \frac{\rho}{1 - \rho^2}}{2\pi}$$

