

Osnove Newtonove mehanike

Def: Afini prostor $\mathcal{A}$ nad vektorskim prostorom $\mathcal{V}$ je množica $\mathcal{A}$ z binarno operacijo

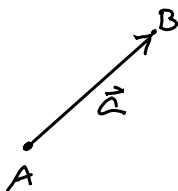
$\mathcal{A} \times \mathcal{V} \rightarrow \mathcal{A} : (A, \vec{a}) \mapsto A + \vec{a}$ z lastnostmi:

i) $(A + \vec{a}) + \vec{b} = A + (\vec{a} + \vec{b})$

ii) $\forall A, B \in \mathcal{A} \exists \vec{a} \in \mathcal{V} : B = A + \vec{a}$

$$\dim \mathcal{A} = \dim \mathcal{V}$$

Primeri:



Def: Definiramo operacijo $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{V}$.

s predpisom $B - A = \vec{a} \Leftrightarrow \vec{B} = A + \vec{a}$

Trditve:

i) $A - A = \vec{0}$

ii) $(A - B) + (B - A) = \vec{0}$

iii) $(A - B) + (B - C) + (C - A) = \vec{0}$

iv) $(A - B) + \vec{a} = (A + \vec{a}) - B$

v) $(A + B) - C = B + (A - C)$

Dokaz:

i) $A - A = \vec{a} \Leftrightarrow A = A + \vec{a} \Leftrightarrow A = (A + \vec{a}) + \vec{c} = A + 2\vec{a}$

$$\Rightarrow \vec{a} = 2\vec{a} \text{ (ker je } \vec{a} \text{ natanko določen)}$$

$$\Rightarrow \vec{a} = \vec{0}$$

v) $A - C = \vec{a} \Rightarrow A = C + \vec{a}$
 $B + (A - C) = B + \vec{a} = C + (\vec{b} + \vec{a})$

$B - C = \vec{b} \Rightarrow B = C + \vec{b}$ velika, zato lahko zamenjamo oklepajo
 $(A + B) - C = ((C + \vec{a}) + (C + \vec{b})) - C =$

$$= C + \vec{a} + (C + \vec{b} - C) = \vec{C} + \vec{a} + \vec{b}$$

vektor

$$(\mathcal{A}, \mathcal{V}) \quad (\mathcal{A}', \mathcal{V}')$$

$$g: \mathcal{A} \rightarrow \mathcal{A}'$$

Def: Preslikava $g: \mathcal{A} \rightarrow \mathcal{A}'$ je **afina** če obstaja

$dg \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ tako da velja $g(A) - g(B) = dg(A - B)$
za $\forall A, B \in \mathcal{A}$

$$g(A) = g(B) + dg(A - B)$$

$$g(A) = g(o) + dg(A - o); \quad o \text{ je } \text{pol afine preslikave}$$

Izbira pola je poljubna

$$\begin{aligned} \tilde{g}(A) &= g(\tilde{o}) + dg(A - \tilde{o}) = g(o) + dg(\tilde{o} - o) + dg(A - \tilde{o}) \\ &= g(o) + dg(\underbrace{(\tilde{o} - o) + (A - \tilde{o})}_{A - o}) = g(A) \end{aligned}$$

$$\text{Izberimo } o \in \mathcal{A} \quad \vec{a} = A - o$$

$$A \in \mathcal{A}; \quad A = o + (A - o) = o + \vec{a}$$

$\forall A \in \mathcal{A}$ lahko identificiramo z vektorjem $\vec{a} \in \mathcal{V}$

Definicija: Galilejeva struktura G je trojica $(\mathcal{A}, \tau, d)$, kjer je $\mathcal{A}$ štirirazsežen afini prostor, $\tau \in \mathcal{L}(V, \mathbb{R})$ in d evklidska razdelje nad

$\uparrow$
 linearna
 presliheva

prostoram istočasnih dogodkov.

Funkcionalu τ pravimo **časovnost**. Dogodka $A, B \in \mathcal{A}$ sta **istočasna**, če $A - B \in \tau^{-1}(0)$

Definicija: Galilejevi strukturi $G(\mathcal{A}, \tau, d)$ in $\tilde{G}(\tilde{\mathcal{A}}, \tilde{\tau}, \tilde{d})$ sta **ekvivalentni**, če obstaja afina bijekcija $g: \mathcal{A} \rightarrow \tilde{\mathcal{A}}$, ki ohranja časovnost in razdelje istočasnih dogodkov

$$\tilde{\tau}(g(A) - g(B)) = \tau(A - B)$$

$$A, B \text{ istočasna} \Leftrightarrow g(A), g(B) \text{ istočasna}$$

$$d(A, B) = \tilde{d}(g(A), g(B))$$

Definicija
 $\mathbb{R} \times E$ afini prostor, kjer je E torazsežni: evklidski prostor

Na $\mathbb{R} \times E$ vpeljemo naravno Galilejevo strukturo.

$$A \in \mathbb{R} \times E \Rightarrow A = (t, P) \quad t \in \mathbb{R}, P \in E$$

$\nwarrow$ ima normo porajeno s skalarnim produktom

$$\tau(A_2 - A_1) = t_2 - t_1$$

$$d(A_1, A_2) = \|P_1 - P_2\|$$

Taj struktur; pravimo **naravna Galilejeva struktura**

Definicija: **Koordinatni sistem** na afinem prostoru $\mathcal{A}$

je bijektivna preslikev $\varphi: \mathcal{A} \rightarrow \mathbb{R}$

$$A \mapsto \varphi(A) = (\pi_t \varphi(A), \pi_P \varphi(A))$$

za katero- φ φ_t afina preslikev $= \varphi_t(A), \varphi_P(A)$

$$(\mathbb{R} \times E, \tau, \|\cdot\|)$$

$$\tau(A - B) = \tau(\varphi_t(A) - \varphi_t(B))$$

$$d(A, B) = \|\varphi_P(A) - \varphi_P(B)\|$$

Lahko moramo razdeljati tudi med neistozanimi dogodki.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi} & \mathbb{R} \times E \\ & \searrow \varphi & \updownarrow \\ & & \mathbb{R} \times E \end{array}$$

Kdaj sta dve naravni galilejevi strukturi ekvivalentni

$$g: \mathbb{R} \times E \rightarrow \mathbb{R} \times E$$

$$A = \begin{bmatrix} t \\ p \end{bmatrix} \mapsto g(A) = \begin{bmatrix} t' \\ p' \end{bmatrix} = g(0) + dg(A-0)$$

$$= \begin{bmatrix} t_0' \\ p_0' \end{bmatrix} + \begin{bmatrix} \alpha & \vec{a}^T \\ \vec{c} & Q \end{bmatrix} \begin{bmatrix} t - t_0 \\ p - p_0 \end{bmatrix}$$

$$0 = \begin{bmatrix} t_0' \\ p_0' \end{bmatrix} \quad \begin{matrix} 3 \times 3 \\ \text{metrika} \end{matrix} \nearrow$$

$$A_1 = \begin{bmatrix} t_1 \\ p_1 \end{bmatrix} \quad A_2 = \begin{bmatrix} t_2 \\ p_2 \end{bmatrix}$$

$$t(g(A_2) - g(A_1)) = t(A_2 - A_1)$$

$$g(A_2) - g(A_1) = \begin{bmatrix} \alpha & \vec{a}^T \\ \vec{c} & Q \end{bmatrix} \begin{bmatrix} t_2 - t_1 \\ p_2 - p_1 \end{bmatrix} = \begin{bmatrix} \alpha(t_2 - t_1) + \vec{a}^T(p_2 - p_1) \\ (t_2 - t_1)\vec{c} + Q(p_2 - p_1) \end{bmatrix}$$

$$\Rightarrow \alpha(t_2 - t_1) + \vec{a}^T(p_2 - p_1) = t_2 - t_1 \Rightarrow \alpha = 1, \vec{a} = \vec{0}$$

To velja da se ohranja časovnost

ohranjanje razdalje med istočasniki, dogodki:

$$d(g(A_1), g(A_2)) = d(A_2, A_1) \quad \text{za } t_1 = t_2$$

$$\parallel \leftarrow t_2 - t_1 = 0$$

$$\parallel$$

$$\parallel Q(p_2 - p_1) \parallel$$

$$\parallel p_2 - p_1 \parallel$$

$$\Rightarrow Q \in O(3)$$

$\leftarrow$ ortogonalna metrika 3×3

Definicija: Preslikava, ki ohranja Galilejevo strukturo pravimo **Galilejeve preslikave**

Trditev: Galilejeve preslikave med marnima Galilejevima strukturama $R \times E$ je oblike

$$\begin{bmatrix} t \\ p \end{bmatrix} \mapsto \begin{bmatrix} t' \\ p' \end{bmatrix} = \begin{bmatrix} t_0' + t - t_0 \\ p_0' + \vec{c}(t - t_0) + Q(p - p_0) \end{bmatrix} =$$

kjer je $Q \in O(3)$, $\vec{c}$ poljuben vektor, t_0 poljubno število in p_0 poljubna točka

$$= \begin{bmatrix} \tilde{t}_0 + t \\ \tilde{p}_0 + \vec{c}t + Q(p - p_0) \end{bmatrix}$$

$\uparrow$
 $t_0' - t_0 = \tilde{t}_0$

Dva apazovalea: $p(t, P)$, $p'(t', P)$

gibanje $t \mapsto P(t)$ *terakcija tocke*

$$\vec{v} = \frac{dP}{dt} \quad \text{vektor brzine} \quad \lim_{h \rightarrow 0} \frac{P(t+h) - P(t)}{h}$$

$$\ddot{p} = \dot{\vec{v}} = \vec{a} = \frac{d\vec{v}}{dt} \quad \text{vektor pospeška}$$

$$|\vec{v}| = v \quad \text{brzina}$$

$$Q(P(t) - P_0)$$

$$P'(t) = P_0' + \vec{c}(t' - t_0') + \overbrace{Q(P(t) - P_0)}^{\text{terakcija v } p'}$$

$$V' = \frac{dP'}{dt'} = \vec{c} + Q \frac{dP}{dt}(t' - t_0') = \vec{c} + QP(t)$$

$$\ddot{p}' \quad \dot{P}'(t') = \vec{c} + Q\dot{P}(t)$$

↑ najprej orbita, nato odvod

$$\ddot{P}'(t') = \vec{a}' = \frac{dV'}{dt'} = Q\ddot{P}(t)$$

φ	φ'
P	$P = P_0' + \vec{c}t + Q(P - P_0)$
$\vec{a} = P_2 - P_1$	$\vec{a}' = P_2' - P_1' = Q(P_2 - P_1) = Q\vec{a}$

Definicija: vektor $a = P_2 - P_1$ je *koordinatno neodvisen*

$$A \in \mathcal{L}(V, V) \quad A' = Q^T A Q$$

$$\lambda \quad \lambda' = \lambda$$

Sistem materialnih točk $(p_1, \dots, p_n) = \underline{p}$

$$\underline{p}' = \underline{p}_0' + \underline{c}t + Q(\underline{p} - \underline{p}_0)$$

$$\underline{p}_0' = (p_0', \dots, p_n')$$

$$Q(\underline{p} - \underline{p}_0) = (Q(p_1 - p_0), \dots, Q(p_n - p_0))$$

$$\underline{c} = (\underline{c}_1, \dots, \underline{c}_n)$$

$$\Rightarrow \underline{\dot{p}}' = \underline{c} + Q \underline{\dot{p}}$$

$$\underline{\ddot{p}}' = Q \underline{\ddot{p}}$$

Princip determiniranosti

V danem KS (koordinatni sistem) je trajektorija sistema materialnih točk natanko deločena z začetnim položajem in začetno hitrostjo.

To specialno pomeni, da obstaja funkcija interakcije $\underline{\vec{f}}$ tako da je $\underline{\ddot{p}} = \underline{\vec{f}}(t, \underline{p}, \underline{\dot{p}})$

$$\left(\ddot{p}(t) = \vec{f}(t, \underline{p}(t), \dot{p}(t)) \text{ nedolgo} \right)$$

Princip relativnosti ^{8.10} obstaja razred

koordinatnih sistemov v katerem je funkcija interakcije invariantna za Galilejeve transformacije. Koordinatnim sistemom iz tega pravimo **inercialni koordinatni sistem**

$$(t, \underline{P}) \xrightarrow{GT} (t', \underline{P}') \text{ Galilejeve transformacije.}$$

$$\text{Potem je } \underline{f} = \underline{f}' \quad (\ddot{\underline{P}} = \ddot{\underline{P}}')$$

$$\ddot{\underline{P}}' = \ddot{\underline{f}}(t', \underline{P}', \dot{\underline{P}}')$$

$$Q \ddot{\underline{P}} = Q \ddot{\underline{f}}(t, \underline{P}, \dot{\underline{P}}) = \ddot{\underline{f}}(t_0' + t, \underline{P}_0' + \vec{c}t + Q(\underline{P} - \underline{P}_0), \vec{c} + Q\dot{\underline{P}})$$

$$i) \quad \vec{c} = \vec{0} \quad Q = I \quad \underline{P}_0' = \underline{P}_0$$

$$\ddot{\underline{f}}(t, \underline{P}, \dot{\underline{P}}) = \ddot{\underline{f}}(t_0' + t, \underline{P}, \dot{\underline{P}}) \text{ za } \forall t_0'$$

$\Rightarrow f_n$ eksplicitno odvisna od čase (t)

(homogenost časa)

$$ii) \quad \vec{c} = \vec{0}, Q = I, \dot{\underline{P}}' = \dot{\underline{P}} + \vec{a}$$

$$\ddot{\underline{f}}(\underline{P}, \dot{\underline{P}}) = \ddot{\underline{f}}(\underline{P} + \vec{a}, \dot{\underline{P}}) \text{ za } \forall \vec{a}$$

$$\parallel$$

$$(\underline{P}_1 + \vec{a}, \dots, \underline{P}_n + \vec{a})$$

$\Rightarrow \ddot{\underline{f}}$ je odvisna samo od relativnih položajev

$$\ddot{\underline{f}}(\underline{P}, \dot{\underline{P}}) = \ddot{\underline{f}}(\underline{P}_i - \underline{P}_j, \dot{\underline{P}})$$

$i \neq j$ t

vse kombinacije tega

(homogenost prostora)

$$iii) \quad Q = I$$

$$f(\underline{P}_i - \underline{P}_j, \dot{\underline{P}}) = f(\underline{P}_i - \underline{P}_j, \vec{c} + \dot{\underline{P}}) \Rightarrow \ddot{\underline{f}}(\underline{P}_i - \underline{P}_j, \dot{\underline{P}}_k - \dot{\underline{P}}_l)$$

(homogenost prostora hitrost)

$$iv) \quad Q \text{ poljuben}$$

$$Q \ddot{\underline{f}}(\underline{P}_i - \underline{P}_j, \dot{\underline{P}}_k - \dot{\underline{P}}_l) = \ddot{\underline{f}}(Q(\underline{P}_i - \underline{P}_j), Q(\dot{\underline{P}}_k - \dot{\underline{P}}_l))$$

$\ddot{\underline{f}}$ je izotropna funkcija

$$Q \ddot{\underline{g}}(\vec{a}) = \ddot{\underline{g}}(Q \vec{a}) \quad \checkmark \quad \forall Q \in O(3)$$

Posebni primer : $N=1$ (izolirana točka)

$$\ddot{\vec{p}} = \vec{f}(\quad) \quad \text{brez argumentov} \quad (\text{konstanta})$$

Se vedno velja izotropno st $Q\ddot{\vec{p}} = Q\vec{f} = \vec{f} \Rightarrow \vec{f} = 0$
 $\Rightarrow \ddot{\vec{p}} = \vec{0}$

v IKS (inercialni koordinatni sistem) se izolirana materialna točka giblje premosoothno s konstantno
brzino. $\vec{p} = \vec{v}_0 t + \vec{p}_0 (t=0)$

$N=2 \Rightarrow$

$$\ddot{\vec{p}}_1 = \vec{f}_1(\vec{p}_1 - \vec{p}_2, \dot{\vec{p}}_1 - \dot{\vec{p}}_2)$$

$$\ddot{\vec{p}}_2 = \vec{f}_2(\vec{p}_2 - \vec{p}_1, \dot{\vec{p}}_2 - \dot{\vec{p}}_1)$$

če se točke gibata po
premici proti drugi točki
vedno osteli na tej premici
(Podobno to: točke ostanejo
na ravni:)

$$\ddot{\vec{p}}_1 = f_1(\vec{p}_1 - \vec{p}_2, \vec{p}_1 - \vec{p}_3, \underline{\vec{p}_2 - \vec{p}_3}, \dots)$$

↑

tudi
to je pomembno

Definicija: Interakcija $\ddot{\vec{P}}_i = f(\dots)$ je *parska* če je odvisna samo od relativnih položajev in hitrosti glede na $\vec{P}_i$ in je delovanje tudi avtonomno

$$\ddot{\vec{P}}_i(\vec{P}_j - \vec{P}_i, \dot{\vec{P}}_j - \dot{\vec{P}}_i)$$

$j \neq i \qquad k \neq i$

$$\ddot{\vec{P}}_i = \sum_{\substack{j=1 \\ j \neq i}}^n \ddot{\vec{P}}_{ji}(\vec{P}_i - \vec{P}_j, \dot{\vec{P}}_i - \dot{\vec{P}}_j)$$

Def: Interakcija $\ddot{\vec{P}}_{ji} = \ddot{\vec{P}}_{ji}(\vec{P}_i - \vec{P}_j, \dot{\vec{P}}_i - \dot{\vec{P}}_j)$ je *lokalna* če velja $\lim_{|\vec{P}_i - \vec{P}_j| \rightarrow \infty} \ddot{\vec{P}}_{ji} = \vec{0}$

Princip sorazmernosti

V IKS za sistem materialnih točk $P_1, \dots, P_N$ obstajajo natanko določene konstante α_{ij} tako, da ne glede na interakcije $\vec{F}_i$ velja $\vec{F}_i = - \sum_{j=1}^N \alpha_{ij} \vec{F}_j$

Lema: Za konstante α_{ij} velja

i) $\alpha_{ij} \alpha_{ji} = 1$

ii) $\alpha_{ij} \alpha_{jk} \alpha_{ki} = 1$

Dokaz:

$$\vec{F}_i = \vec{F}_i (P_i, -P_N) = \vec{F}_i (P_i, -P_1, P_i, -P_2, \dots, P_i, -P_N)$$

in naj bodo P_i lokalni

P_N pošljemo proti ∞ razen P_i in P_j

$$\vec{F}_i = - \sum_{\substack{k=1 \\ k \neq i}}^N \alpha_{ki} \vec{F}_k = -\alpha_{ji} \vec{F}_j$$

$$\vec{F}_j = - \sum_{\substack{k=1 \\ k \neq j}}^N \alpha_{kj} \vec{F}_k = -\alpha_{ji} \vec{F}_i$$

$$F = -\alpha_{ji} \vec{F}_j = \alpha_{ji} \alpha_{ij} \vec{F}_i \Rightarrow \alpha_{ji} \alpha_{ij} = 1$$

P_i, P_j, P_N ostanejo, ostale gredo proti ∞

$$\begin{aligned} \vec{F}_i &= -\alpha_{ji} \vec{F}_j - \alpha_{ki} \vec{F}_k = -\alpha_{ji} (-\alpha_{ij} \vec{F}_i - \alpha_{kj} \vec{F}_k) - \alpha_{ki} \vec{F}_k = \\ \vec{F}_i &= \alpha_{ij} \vec{F}_i - \alpha_{kj} \vec{F}_k \end{aligned}$$

$$\vec{F}_i = \underbrace{\alpha_{ji} \alpha_{ij}}_1 \vec{F}_i + \alpha_{ji} \alpha_{kj} \vec{F}_k - \alpha_{ki} \vec{F}_k$$

$$\alpha_{ki} \vec{F}_k = \alpha_{ji} \alpha_{kj} \vec{F}_k / \alpha_{ik}$$

$$\vec{F}_k = \alpha_{ji} \alpha_{kj} \alpha_{ik} \vec{F}_k \Rightarrow \alpha_{ik} \alpha_{kj} \alpha_{ji} = 1$$

Lema: Naj za pozitivna števila α_{ij} velja

i) $\alpha_{ij} \cdot \alpha_{ji} = 1$

ii) $\alpha_{ij} \alpha_{jk} \alpha_{ki} = 1$.

Potem $\exists$ pozitivna števila m_i , tako da velja da je

$$\alpha_{ji} = \frac{m_j}{m_i}$$

Števila m_i so določena do sorazmernostnega

faktorja natančnost. Številom m_i pravimo

inersijske mase

Dokaz: $\alpha_{ii} = 1$

$$\alpha_{ij} \alpha_{ji} \alpha_{ji} = 1 \Rightarrow \alpha_{ji} = 1$$

$$l_{ij} = \log \alpha_{ij} \quad l_{ij} + l_{jk} + l_{ki} = 0$$

$$l_{ij} = -l_{ji}$$

$$l_{i_0 j} + l_{jk} + l_{k i_0} = 0$$

$$l_{ij} - l_{i_0 j} + l_{ki} - l_{k i_0} = 0 \Rightarrow$$

$$l_{ij} - l_{i_0 j} = -l_{ki} + l_{k i_0} = l_{i_0 i} - l_{i i_0} \quad \forall j, k$$

$$\Rightarrow l_{ij} - l_{i_0 j} = n_{i i_0}$$

$$\left. \begin{aligned} l_{ij} &= n_{i i_0} + l_{i_0 j} \\ j &= 1 \end{aligned} \right\} \Rightarrow 0 = n_{i i_0} + l_{i_0 i} \Rightarrow l_{i i_0} = n_{i i_0}$$

$$l_{ij} = l_{i i_0} - l_{j i_0}$$

$$l_{i i_0} = \log m_i \leftarrow \text{definiram}$$

$$\Rightarrow l_{ij} = \log m_i - \log m_j = \log \frac{m_i}{m_j} \Rightarrow \alpha_{ij} = \frac{m_i}{m_j}$$

Izrek: V inercialnem koordinatnem sistemu velja

$$m_1 \ddot{\vec{P}}_1 + m_2 \ddot{\vec{P}}_2 + \dots + m_n \ddot{\vec{P}}_n = \vec{0}$$

Dokaz:

$$\vec{F}_i = - \sum_{\substack{j=1 \\ j \neq i}}^N \alpha_j \cdot \vec{F}_j = - \sum_{j=1}^N \frac{m_j}{m_i} \vec{F}_j$$

$$\ddot{\vec{P}}_i = - \sum_{j=1}^N \frac{m_j}{m_i} \ddot{\vec{P}}_j \quad / m_i$$

$$\sum_{i=1}^N m_i \ddot{\vec{P}}_i = \vec{0}$$

$$\vec{P}_* = \frac{1}{m} \sum_{i=1}^N m_i \vec{P}_i, \quad m = \sum_{i=1}^N m_i$$

masno središče

seštevanje točk:

$$\sum_{i=1}^N \vec{P}_i = \sum_{i=1}^N (m_i (\underbrace{\vec{P}_i - \vec{O}}_{\text{vektor}}) + m_i \vec{O}) = \sum_{i=1}^N m_i (\vec{P}_i - \vec{O}) + \underbrace{\sum_{i=1}^N m_i \vec{O}}_{\vec{O}}$$

$$\vec{P}_* = \vec{O} + \frac{1}{m} \sum_{i=1}^N m_i (\vec{P}_i - \vec{O})$$

$$m \ddot{\vec{P}}_* = \vec{0}$$

Sila $n = P_i$
 Definicija: je produkt interakcije k z maso m_i
 $m_i \ddot{P}_i = m_i f(\dots) =: \vec{F}_i$

Princip o masi masa materialnih tade je enaka
 v vseh koordinatnih sistemih

$$\ddot{P}_1 = f_1(P_1 - P_2, \dot{P}_1 - \dot{P}_2) \quad m_1 \ddot{P}_1 = F_1$$

$$\ddot{P}_2 = f_2(P_1 - P_2, \dot{P}_1 - \dot{P}_2) \quad m_2 \ddot{P}_2 = F_2$$

$$\overline{m_1 \ddot{P}_1 + m_2 \ddot{P}_2} = F_1 + F_2 = 0$$

$$F_2 = -F_1$$

$$\vec{F}_1 = \vec{F}_{21}$$

$$F_2 = F_{12}$$

$$\vec{F}_{12} = -\vec{F}_{21}$$

Trditev: Če so vse sile parske in lokalne velja zakon akcije in reakcije (3.12)

$$\text{Dokaz: } \vec{A}_i = \sum_{\substack{j=1 \\ j \neq i}}^N \vec{f}_{ji} (p_j - p_i, \dot{p}_j - \dot{p}_i)$$

$$\downarrow$$

$$\{ p_j \rightarrow \infty \quad j \in \{1, \dots, k\} \}$$

$$m_i \vec{f}_i = m_i \ddot{\vec{p}}$$

$$m_i \vec{f}_{ji} + m_j \vec{f}_{ij} = \vec{0}$$

$$\vec{F}_{ji} = -\vec{F}_{ij}$$

Y

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~~amade~~ prva wo

odvod vzdolž traktorijske $\frac{d}{dt} \in (t, p(t), \dot{p}(t)) =$

$$\frac{\partial E}{\partial t} + \left(\frac{\partial E}{\partial p} \right)^T \dot{p} + \left(\frac{\partial E}{\partial \dot{p}} \right)^T \ddot{p}$$

Izrek o množenju: Naj bo sila $\vec{F}$ potencialna s potencialom $U(t, p, \dot{p})$. Potem je vsota kinetične in potencialne energije konstanta

gibanja (= njen odvod vzdolž traktorijske je 0)

$\Leftrightarrow$ moč sil je nasprotna enaka odvodu potencialne energije vzdolž traktorijske

Dedz: $\Rightarrow E_0 = T + U = \frac{1}{2} m |\dot{p}|^2 + U$

$$0 = \frac{\partial E_0}{\partial t} = m \ddot{p} + \dot{p} \ddot{p} + \frac{dU}{dt} \Rightarrow \frac{dU}{dt} = -\dot{p} \ddot{p}$$

$\Leftarrow$

$$\overset{U_1}{T_2} - \overset{U_1}{T_1} = \int_{t_1}^{t_2} \dot{\vec{r}} \cdot \ddot{\vec{p}} dt = - \int_{t_1}^{t_2} \frac{dU}{dt} dt = U(t_1) - U(t_2)$$

$$\Rightarrow U_1 + T_1 = U_2 + T_2 = E_0$$

Sila je **konzervativna**, če je potencialna in
odvisna samo od položaja

Posledica: če je sila konzervativna velja izrek
o energiji

$$\text{Dokaz: } \frac{d}{dt} U(\mathbf{p}(t)) = \left(\frac{\partial U}{\partial \mathbf{p}} \right)^T \dot{\mathbf{p}} = -\mathbf{F} \dot{\mathbf{p}}$$

$$\mathbf{F} \dot{\mathbf{p}} = -\frac{dU}{dt}$$

$$\mathbf{F} \dot{\mathbf{p}} = Q \mathbf{F}(\dot{\mathbf{c}} + Q \dot{\mathbf{p}}) = \underbrace{Q \mathbf{F} \cdot \dot{\mathbf{c}}}_{\substack{\parallel \\ \mathbf{F} \cdot \dot{\mathbf{c}}}} + \underbrace{Q \mathbf{F} \cdot Q \dot{\mathbf{p}}}_{\mathbf{F} \cdot \dot{\mathbf{p}}} = -\frac{dU}{dt} + m \ddot{\mathbf{p}} \cdot \dot{\mathbf{c}} =$$

$$= -\frac{d}{dt} (U - m \dot{\mathbf{p}} \cdot \dot{\mathbf{c}})$$

↑
artemidova
preoblikovna
skalarni produkt

$$\tilde{U} = U - m \dot{\mathbf{p}} \cdot \dot{\mathbf{c}}$$

$$\tilde{U}(\mathbf{p}, \dot{\mathbf{p}}) = U(\mathbf{p}) - m \dot{\mathbf{p}} \cdot \dot{\mathbf{c}}$$

Izrek: Izrek o energiji je invarianten za Galilejeve transformacije

$$\begin{aligned}
 t_0' &= T' + \tilde{U}' = \frac{1}{2} m |\vec{c}|^2 + m c Q \dot{P} + T + \tilde{U} - m \dot{P}' \cdot \vec{c} = \\
 &= E_0 + \frac{1}{2} m |\vec{c}|^2 + m \vec{c} (Q \dot{P} - \dot{P}') = E_0 - \frac{1}{2} m |\vec{c}|^2 \\
 T' &= \frac{1}{2} m \dot{P}' \cdot \dot{P}' = \frac{1}{2} m (\vec{c} + Q \dot{P}) \cdot (\vec{c} + Q \dot{P}) = \frac{1}{2} m |\vec{c}|^2 + m \vec{c} \cdot Q \dot{P} \\
 &\quad + \frac{1}{2} m |\dot{P}|^2 \\
 \dot{P}' &= \vec{c} + Q \dot{P}
 \end{aligned}$$

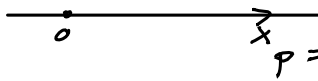
Premočno gibanje

= pospešek ima konstantno smer

$$\vec{a} = a(t) \vec{e}$$

$$\vec{v} = \int_{t_0}^t \vec{a} dt + \vec{v}(t_0) = \left(\int_{t_0}^t a(t) dt \right) \vec{e} + \vec{v}(t_0)$$

Obstaja KS v katerem tir leži na premici



$$p = x \quad \dot{p} = \dot{x} \quad \ddot{p} = \ddot{x}$$

$$m\ddot{x} = f(t, x, \dot{x})$$

$v = \dot{x}$ je lahko edaj tudi
negativen

omejimo se ko bo f konzervativna

$$m\ddot{x} = f(x)$$

kde je sila v odvisnosti položaja potencialna?

(potencialna $\Rightarrow$ konzervativna)

$$f = -\frac{dU}{dx} \Rightarrow U = \int_{x_0}^x f(\xi) d\xi + U_0$$

sila $f(x)$ je potencialna če je f zvezna

če je $f(x)$ zvezna $\Rightarrow$ velja izrek o energiji

$$m\ddot{x} = F(x)$$

$$\frac{1}{2}m\dot{x}^2 + U(x) = E_0$$

$$\dot{x}^2 = \frac{2}{m}(E_0 - U(x))$$

$$\frac{dx}{dt} = \dot{x} = \pm \sqrt{\frac{2}{m}(E_0 - U(x))}$$

$$\pm \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}} = dt$$

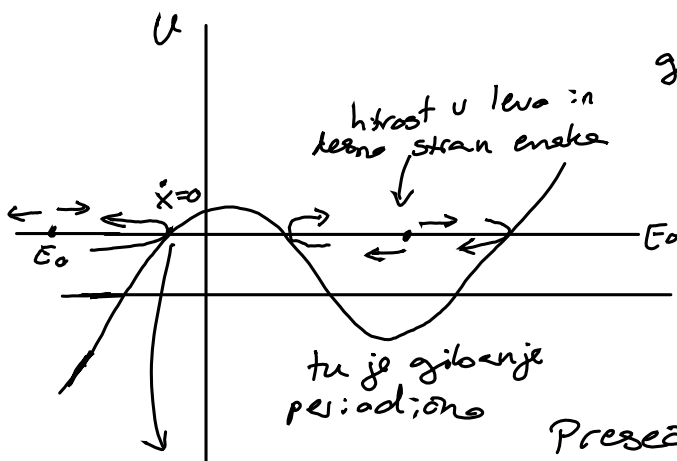
$$\text{sgn}(\dot{x}) \int_{x_0}^x \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}} = \int_{t_0}^t dt = t - t_0 \Rightarrow t = T(x) \Rightarrow x = x(t)$$

↑

lahko beremo
ko $x \neq 0$
($\text{sgn}(\dot{x})$ je
konstanten)

$$t = t_0 \pm \int_{x_0}^x \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$$

Kvali: kvalitna obravnava gibanja



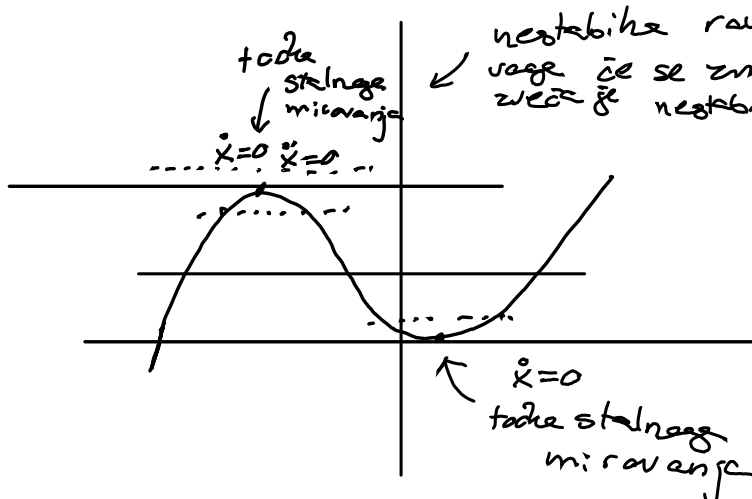
gibanje je možno
tako kjer je

energija nivojska
nad potencialom
 E_0 nad grafom

Presečišče je točka

trenutnega miravanja
(točka obrate gibanja)

$$0 = \dot{x} = \frac{F(x)}{m} = -\frac{dU}{dx}$$

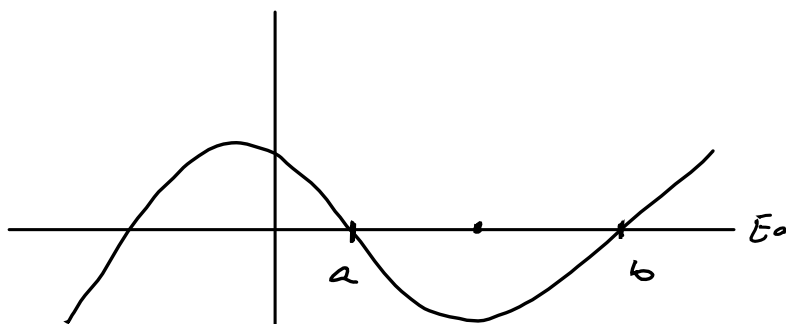


nestabilna ravnovesna
lega če se zmanjša ali
zveča je nestabilna

Lokalni minimum je stabilna ravnovesna lega
(če se energija malo spremeni pride do majhnega
periodičnega gibanja (mehanika o d. ravnovesne
lege))

Prevoj je tudi nestabilna lega

Kakšna je perioda periodičnega gibanja



gibanje: $t_1 = \int_{x_1}^b \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$ $t_2 = -\int_a^b \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$

$t_3 = \int_a^{x_0} \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$

$T = t_1 + t_2 + t_3 = 2 \int_a^b \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}} = \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E_0 - U(x)}}$

↑
perioda

Ali je res periodično in se ne ustavi prej?

atd

$\int_a^{\infty} \frac{dx}{\sqrt{E_0 - U(x)}}$

$U(x) = U(a) + \left(\frac{dU}{dx}\right)\left(\frac{1}{2}\right)(x-a)$

$\frac{dU}{dx}(a) < 0$ a je presecišče

$\exists \delta > 0, \forall \epsilon \in (a, a+\delta) \Rightarrow 2 \frac{dU}{dx}\left(\frac{1}{2}\right) < \frac{1}{2} \frac{dU}{dx}(a)$

$E_0 - U(x) = -\frac{dU}{dx}\left(\frac{1}{2}\right)(x-a) < -2 \frac{dU}{dx}(a)(x-a)$

$\Rightarrow$ integral je konvergenten

ker $\int_a^{\infty} \frac{dx}{\sqrt{E_0 - U(x)}} \leq \frac{1}{\sqrt{-\frac{1}{2}U'(x)(a)}} \int_a^{\infty} \frac{dx}{\sqrt{x-a}} < \infty$

Če se energijske nivojnica dotika lokalnega maksimuma, potem ne pridemo do dotikalnice

v končnem času

Lema $\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}} = \pi$

Dokaz: $x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos\vartheta$

$$b-x = \frac{1}{2}(b-a)(1-\cos\vartheta)$$

$$x-a = \frac{1}{2}(b-a)(1+\cos\vartheta)$$

$$(b-x)(x-a) = \left(\frac{1}{2}(b-a)\right)^2 (1-\cos^2\vartheta) = \overset{\sin^2\vartheta}{\sqrt{\left(\frac{1}{2}(b-a)\right)^2}}$$

$$dx = -\frac{1}{2}(b-a)\sin\vartheta d\vartheta$$

$$I = - \int_0^\pi \frac{\frac{1}{2}(b-a)\sin\vartheta d\vartheta}{\frac{1}{2}(b-a)\sin\vartheta} = - \int_0^\pi d\vartheta = \pi$$

Primer:

Harmonični oscilator $U = \frac{1}{2}kx^2$

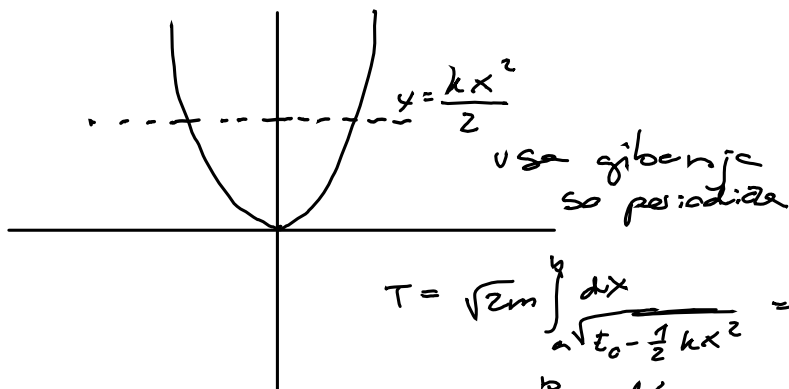
$$F = -kx$$

$$m\ddot{x} = -kx$$

$$\ddot{x} + \frac{k}{m}x = 0 \quad \frac{k}{m} = \omega^2$$

$$\ddot{x} + \omega^2 x = 0$$

$$x = A \cos(\omega t - \delta)$$



$$T = \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E_0 - \frac{1}{2}kx^2}} =$$

$$= \frac{\sqrt{2m}}{\sqrt{\frac{1}{2}k}} \int_a^b \frac{dx}{\sqrt{2E_0 - kx^2}} = 2\sqrt{\frac{m}{k}} \pi = \frac{2\pi}{\omega}$$

T perioda je neodvisna od energije
(izokronono gibanje)

Primer: $U = ax^2 + bx^{-2}$

$[U]$ dimenzija energije

$$[U] = [ax^2] = [a][x^2] = [a] \overset{\text{enota}}{\downarrow} L^2$$

$$[U] = [bx^{-2}] = [b] L^{-2}$$

$$\Rightarrow [a] L^2 = [b] L^{-2}$$

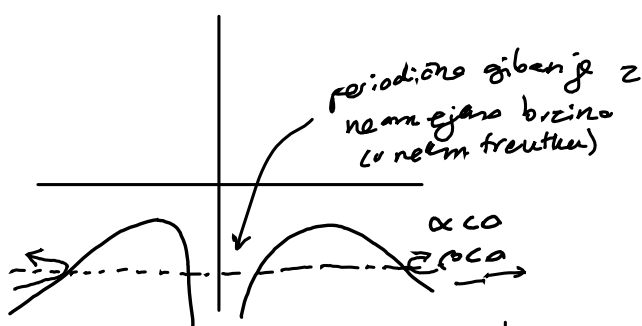
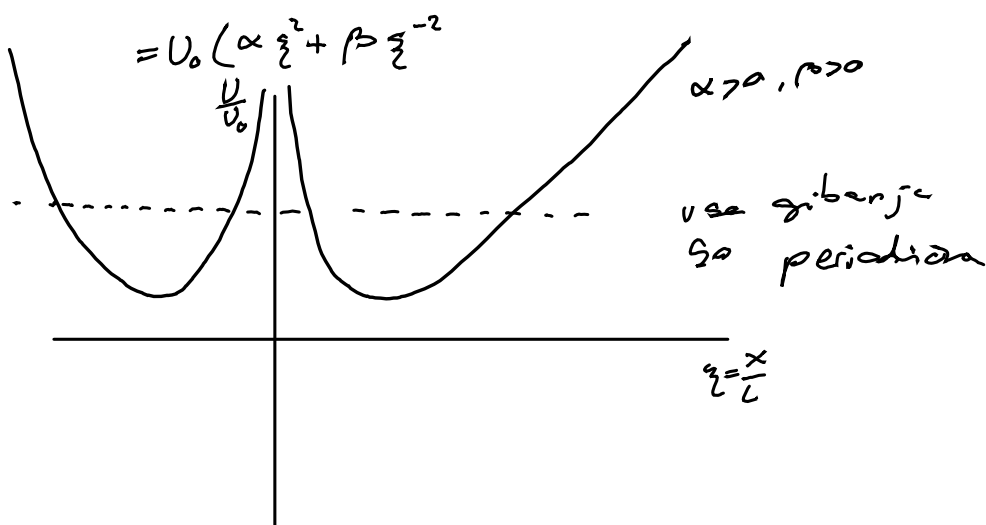
$$L^4 = \frac{[b]}{[a]} \Rightarrow L = \sqrt[4]{\frac{[b]}{[a]}}$$

$$x = L \xi \quad L = \left(\frac{b}{a}\right)^{\frac{1}{4}}$$

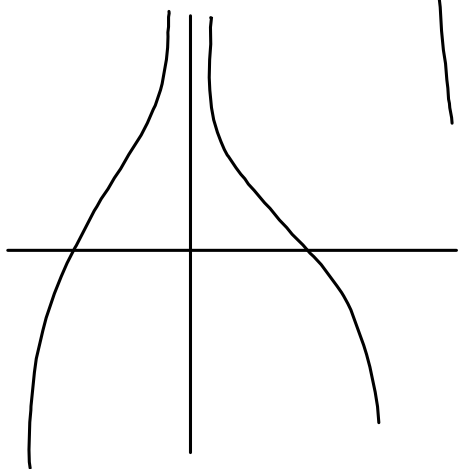
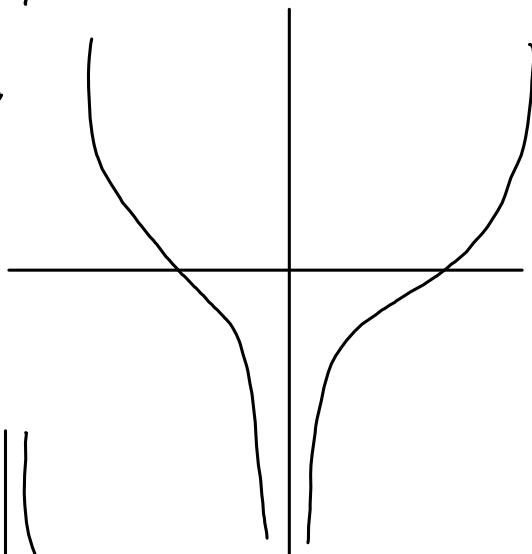
τ brezdimenzijski

$$U = a \sqrt{\frac{|b|}{|a|}} \xi^2 + b \sqrt{\frac{|a|}{|b|}} \xi^{-2} = \underbrace{\sqrt{|a||b|}}_{U_0} (\alpha \xi^2 + \beta \xi^{-2}) =$$

$$a = \underbrace{\frac{a}{|a|}}_{\alpha} |a| \quad b = \underbrace{\frac{b}{|b|}}_{\beta} |b|$$



$$\alpha > 0, \beta < 0$$



Računajmo periode za $\alpha, \beta > 0$

$$U = \sqrt{|a||b|} (\xi^2 + \xi^{-2})$$

U_0 ← *stevilko*
 $E_0 = U_0 \epsilon_0$

$$T = \sqrt{2m} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E_0 - (ax^2 + bx^{-2})}} = \sqrt{2m} \int_{\xi_1}^{\xi_2} \frac{L d\xi}{\sqrt{U_0 \epsilon_0 - U_0 (\xi^2 + \xi^{-2})}} =$$

$$= \sqrt{2m} \frac{L}{\sqrt{U_0}} \int_{\xi_1}^{\xi_2} \frac{d\xi}{\sqrt{\epsilon_0 - \xi^2 - \xi^{-2}}} d\xi = \frac{\sqrt{2m}}{\sqrt{U_0}} L \int_{\xi_1}^{\xi_2} \frac{\xi d\xi}{\sqrt{\xi^2 \epsilon_0 - \xi^4 - 1}}$$

$$u = \xi^2$$

$$= \frac{\sqrt{2m}}{\sqrt{U_0}} L \int_{\xi_1^2}^{\xi_2^2} \frac{\frac{1}{2} du}{\sqrt{-u^2 + u \epsilon_0 - 1}} = \sqrt{2m} \frac{L}{\sqrt{U_0}} \frac{1}{2} \pi$$

↑
 v krajščih jima
 imaginarni
 kvadratni koren
 $2 \epsilon_0 - 1$

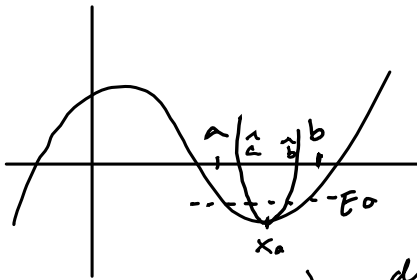
$$U_0 = \sqrt{|a||b|}$$

$$\frac{L}{\sqrt{U_0}} = \frac{|b|^{\frac{1}{4}}}{|a|^{\frac{1}{4}} |a|^{\frac{1}{2}} |b|^{\frac{1}{4}}} = 2 \sqrt{\frac{1}{|a|}}$$

$$T = \sqrt{\frac{m}{2|a|}} \pi \quad \text{spet je izokronično (neodvisno od energije)}$$

in notri ni b

Harmonična optimizacija periode



$$T = \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E_0 - U(x)}}$$

= 0 ker je lok. minimum.

$$U(x) = U(x_0) + \frac{dU}{dx}(x_0)(x - x_0) + \frac{1}{2} \frac{d^2U}{dx^2}(x_0)(x - x_0)^2$$

$$T \approx \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E_0 - U_0 - \frac{1}{2} U''(x_0)(x - x_0)^2}} =$$

$$= \sqrt{2m} \frac{1}{\sqrt{\frac{1}{2} U''(x_0)}} T = 2\pi \sqrt{\frac{m}{U''(x_0)}}$$

$$E_0 = U(x) + \frac{1}{2} m g(x) \dot{x}^2$$

$$u(x) = \int_{x_0}^x \sqrt{g(x)} dx$$

$$\dot{u} = \dot{x} \sqrt{g(x)}$$

$$u^2 = \dot{x}^2 g(x)$$

$$\frac{1}{2} m \dot{u}^2 + U(x) = E_0$$

$$\hat{U}(u) = U(x(u))$$

$$U(x) = \hat{U}(u) \quad \text{with } u = \int \sqrt{g(x)} dx$$

$$U'(x) = \frac{d\hat{U}}{du} \frac{du}{dx} \quad \left. \frac{d\hat{U}}{du} \right|_{u_0} = 0$$

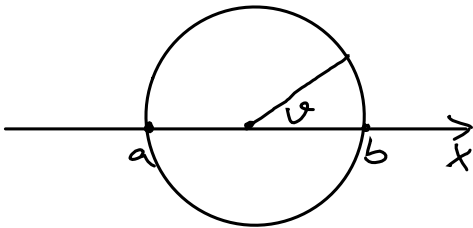
$$U''(x_0) = \frac{d^2\hat{U}}{du^2} \left(\frac{du}{dx} \right)^2 + 0 = \frac{d^2\hat{U}}{du^2}(u_0) \cdot g(x_0)$$

$\leadsto$ harmonische Approximation für

$$T = 2\pi \sqrt{\quad}$$

Libracijska aproksimacija periode

$$t - t_0 = \text{sgn}(\dot{x}) \int_{x_0}^x \frac{dx}{\sqrt{(E_0 - U(x)) \cdot \frac{2}{m}}}$$



$$x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos\psi$$

$$E_0 - U(x) = (b-x)(x-a)\chi(x)$$

$$b-x = \frac{1}{2}(b-a)(1-\cos\psi)$$

$$x-a = \frac{1}{2}(b-a)(1+\cos\psi)$$

← vemo da me n:zle
a in b, odakle
p- se $\chi(x)$

$$= \text{sgn}\dot{x} \int \frac{-\frac{1}{2}(b-a)\sin\psi d\psi}{\sqrt{\left(\frac{1}{2}(b-a)\right)^2 \sin^2 \chi(\psi)}}$$

$$\leq -\text{sgn}(\dot{x}) \int_{\psi_0}^{\psi} \frac{\sin\psi d\psi}{|\sin\psi| \sqrt{\chi(\psi)}}$$

$$\psi \in (0, \pi) \Rightarrow \dot{x} < 0, \sin\psi > 0$$

$$= \int_{\psi_0}^{\psi} \frac{d\psi}{\sqrt{\chi(\psi)}}$$

$$\psi \in (\pi, 2\pi) \Rightarrow \dot{x} > 0 \sin\psi < 0$$

$$= \int_{\psi_0}^{\psi} \frac{d\psi}{\sqrt{\chi(\psi)}}$$

$$t - t_0 = \sqrt{\frac{m}{2}} \int_{\psi_0}^{\psi} \frac{d\psi}{\sqrt{\chi(\psi)}}$$

$$\Rightarrow T = 2\sqrt{\frac{m}{2}} \int_0^{\pi} \frac{d\psi}{\sqrt{\chi(\psi)}} \stackrel{\text{trapezna formula}}{=} \dots$$

$$\approx 2\sqrt{\frac{m}{2}} \frac{1}{2}\pi \left(\frac{1}{\sqrt{\chi(a)}} + \frac{1}{\sqrt{\chi(b)}} \right) = \pi\sqrt{\frac{m}{2}} \left(\frac{1}{\sqrt{\chi(a)}} + \frac{1}{\sqrt{\chi(b)}} \right)$$

$$U(a) = E_0$$

$$U(b) = E_0$$

$$\chi(x) = \frac{E_0 - U(x)}{(b-x)(x-a)}$$

$$\chi(a) = \lim_{x \rightarrow a} \chi(x) = -\frac{U'(a)}{b-a}$$

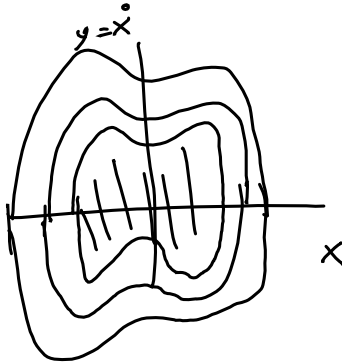
$$\chi(b) = \frac{U'(b)}{b-a}$$



$$T = \pi \sqrt{\frac{m}{2}} \sqrt{b-a} \left(\frac{1}{\sqrt{-U'(a)}} + \frac{1}{\sqrt{U'(b)}} \right) \approx 2\pi \sqrt{\frac{m}{U''(x_0)}}$$

$$U'(a) = U'(x_0) + U''(x_0)(b-a)$$

Izračun: potred



$$\frac{1}{2} m \dot{x}^2 + U(x) = E_0 \Rightarrow$$

$$\frac{1}{2} m y^2 + U(x) = E_0$$

nivojnice
u formi ravnni

$$y = \pm \sqrt{\frac{2}{m} (E_0 - U(x))}$$

$$\frac{dy}{dx} = \pm \sqrt{\frac{2}{m}} \cdot \frac{1}{2} \frac{1 - (-U'(x))}{\sqrt{E_0 - U(x)}}$$

↓ plošćine

$$A(E) = 2 \int_{a(E)}^{b(E)} \sqrt{\frac{2}{m} (E_0 - U(x))} dx$$

ustavimo
b u integralu a i rona

$$\frac{dA}{dE} = 2 b'(E) \cdot 0 - 2 a'(E) \cdot 0 + 2 \sqrt{\frac{2}{m}} \int_a^b \frac{1}{2} \frac{dx}{\sqrt{E - U(x)}} = \frac{1}{m} T$$

$$\Rightarrow T = m \frac{dA}{dE}$$

$$T = \int_a^b \sqrt{2m} \frac{dx}{\sqrt{E - U(x)}}$$

Primer: Harmonični oscilator

$$U = \frac{1}{2} k x^2 \text{ (hookov potencijal)}$$

$$\frac{1}{2} m v^2 + \frac{1}{2} k x^2 = E_0$$

$$\left(\frac{v^2}{\sqrt{\frac{2E_0}{m}}} \right)^2 + \frac{x^2}{\sqrt{\frac{2E_0^2}{k}}} = 1$$

dip se ~~zad.~~
s položajem

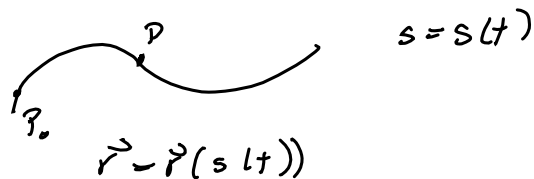
plastična
↓

$$A = \pi \cdot \frac{2E_0}{\sqrt{k m}}$$

$$\Rightarrow \frac{dA}{dE_0} = \frac{2\pi}{\sqrt{k m}} \Rightarrow T = 2\pi \sqrt{\frac{m}{k}}$$

izolaciona
perioda

Gibanje po krivulji



$$\vec{v} = \frac{d\vec{s}}{dt} = \left(\frac{dr}{ds}\right) \dot{s} = \dot{s} \vec{e}_t \quad |\vec{v}| = |\dot{s}|$$

" $\vec{e}_r$

$$\vec{a} = \frac{d}{dt} (\dot{s} \vec{e}_t) = \ddot{s} \vec{e}_t + \dot{s} \frac{d\vec{e}_t}{ds} = \ddot{s} \vec{e}_t + \gamma \vec{e}_n$$

$$\frac{d\vec{e}_t}{ds} = \gamma \vec{e}_n$$

$$\frac{d\vec{e}_t}{ds} \perp \vec{e}_t$$

$$\frac{d\vec{e}_t}{ds} = \frac{\frac{d\vec{e}_t}{dt}}{\frac{ds}{dt}} = \frac{\frac{d\vec{e}_t}{dt}}{|\frac{d\vec{e}_t}{dt}|} \left| \frac{d\vec{e}_t}{dt} \right|$$

krivljenost

$$\gamma = \frac{1}{R} \quad R \dots \text{krivlinski polmer}$$

← normalni vektor

$$|\vec{v} \times \vec{a}| = \gamma |\dot{s}|^3 \Rightarrow \gamma = \frac{|\vec{v} \times \vec{a}|}{|\dot{s}|^3}$$

$$\vec{a} = \ddot{s} \vec{e}_t + \gamma \dot{s}^2 \vec{e}_n \quad \text{tangenti + normalni pospešek}$$

$$\vec{e}_b = \vec{e}_t \times \vec{e}_n \quad \dots, \text{binormalna}$$

$$m\vec{a} = \vec{F} \quad m\vec{a} = \vec{F} + \vec{S} \quad \begin{array}{l} \nearrow \text{rezultante} \\ \text{aktivnih sil} \end{array} \quad \nwarrow \text{rezultanta vee}$$

$$m\vec{S} = \vec{F}\vec{e}_t + \vec{S}\vec{e}_t$$

$$m\vec{S} = \vec{F}\vec{e}_n + \vec{S}\vec{e}_n \quad + \text{konstruiramo na} \\ \text{zvezo sile vee}$$

$$0 = \vec{F}\vec{e}_b + \vec{S}\vec{e}_b$$

Gibanje pokrivajo brez trenja: $\vec{S} \cdot \vec{e}_t = 0$

Idealna krivulja, gladka krivulja

$$\vec{S} \cdot \vec{e}_t = -\frac{\vec{v}}{|\vec{v}|} \cdot \sqrt{|\vec{S} \cdot \vec{e}_n|^2 + |\vec{S} \cdot \vec{e}_s|^2} \cdot \vec{e}_t$$

Gibanje po gladki krivulji

$$\vec{s} \cdot \vec{e}_t = 0$$

$$m\ddot{\vec{s}} = \vec{F} \cdot \vec{e}_x$$

$$F = -\text{grad} U$$

$$U(s) = U(\vec{r}(s))$$

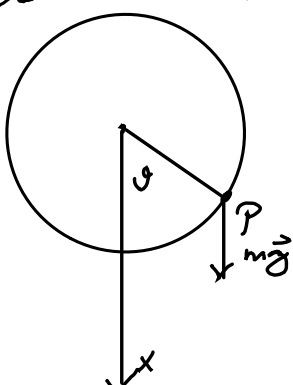
$$\frac{dU}{ds} = \text{grad} U \frac{d\vec{s}}{ds} = -\vec{e}_t$$

$$m\dot{\vec{s}} \cdot \vec{s} = \vec{F} \cdot \vec{e}_x \cdot \vec{s} \rightarrow \frac{1}{m} \dot{\vec{s}}^2 + U(s) = E_0$$

gibanje po gladki krivulji je integrabilno

Primer:

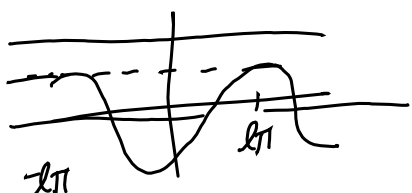
Matematično nihalo = gibanje po gladki krivulji pod vplivom sile teže



$$\vec{F}_g = m\vec{g}$$

$$U = -m\vec{g} \cdot \vec{r} = -m g l (\cos \varphi \vec{i} + \sin \varphi \vec{j}) = -m g l \cos \varphi$$

$$s = l\varphi \quad U(s) = -m g l \cos\left(\frac{s}{l}\right)$$



Reiada matematičnega nihanja

$$T = \sqrt{2m} \int_{-s_0}^{s_0} \frac{ds}{\sqrt{E_0 - U(s)}} = \quad E_0 = -m g l \cos\left(\frac{s_0}{l}\right)$$

$$= 2\sqrt{2m} \frac{1}{\sqrt{mgl}} \int_0^{s_0} \frac{ds}{\sqrt{4 \cos\left(\frac{s}{l}\right) - \cos\left(\frac{s_0}{l}\right)}} \quad s = l\varphi$$

$$= \frac{2\sqrt{2l}}{\sqrt{g}} \int_0^{\varphi_0} \frac{l d\varphi}{\sqrt{\cos \varphi - \cos \varphi_0}} = 2\sqrt{2} \sqrt{\frac{l}{g}} \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{\cos \varphi - \cos \varphi_0}}$$

$$1 = \cos^2 \frac{\varphi}{2} + \sin^2 \frac{\varphi}{2}$$

$$\cos \varphi = \cos^2 \frac{\varphi}{2} - \sin^2 \frac{\varphi}{2}$$

$$1 - \cos \varphi = 2 \sin^2 \frac{\varphi}{2}$$

$$= 2\sqrt{2} \sqrt{\frac{l}{g}} \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{2 \sin^2 \frac{\varphi}{2} - 2 \sin^2 \frac{\varphi_0}{2}}} =$$

$$u = \frac{\sin \frac{\varphi}{2}}{\sin \frac{\varphi_0}{2}}$$

$$\sin^2 \frac{\varphi}{2} = \sin^2 \frac{\varphi_0}{2} u^2$$

$$du = \frac{1}{\sin \frac{\varphi_0}{2}} \cos \frac{\varphi}{2} \cdot \frac{1}{2} d\varphi$$

$$= 2\sqrt{\frac{l}{g}} \int_0^1 \frac{2 du \cdot \sin \frac{\varphi_0}{2}}{\cos \frac{\varphi}{2} \sqrt{\sin^2 \frac{\varphi_0}{2} (1 - u^2)}} =$$

$$= 4\sqrt{\frac{l}{g}} \int_0^1 \frac{du}{\sqrt{1 - \sin^2 \frac{\varphi_0}{2} u^2} \sqrt{1 - u^2}} =$$

$$4\sqrt{\frac{l}{g}} \int_0^1 \frac{du}{\sqrt{1 - k^2 u^2} \sqrt{1 - u^2}} = K(k^2)$$

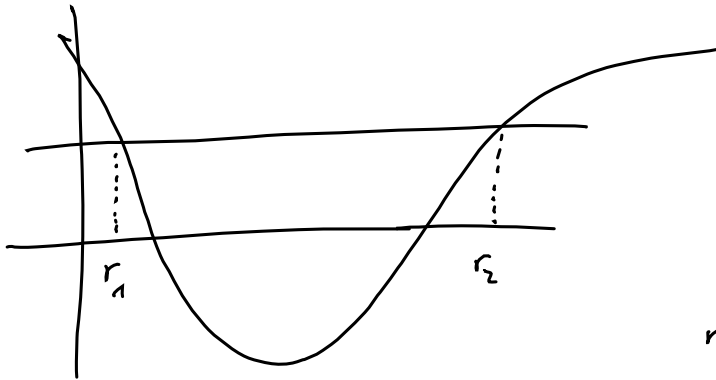
to je popolni eliptični integral prve vrste

$$T = 4\sqrt{\frac{l}{g}} K(k^2)$$

5.10

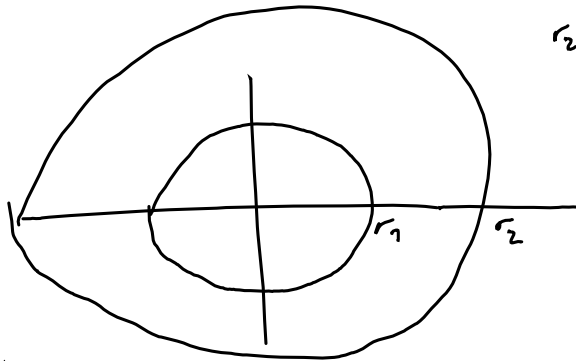
✓

ene wa + ene wa-pakusarje



r_1 pericenter

r_2 apocenter



$$\vec{r} = \vec{r}(v)$$

$$\frac{d\vec{r}}{dt} = \dot{r} \vec{e}_r + r \dot{\varphi} \vec{e}_\varphi$$

Torej se letišče obnaša
krogič

...

tir je sinetroiden

Mege de Filipari zpidu 2??

Relativno gibanje

koordinatni sistem $\varphi(t, P)$ se giblje glede na koordinatni sistem $\varphi'(t', P')$, če $\exists$ trajica

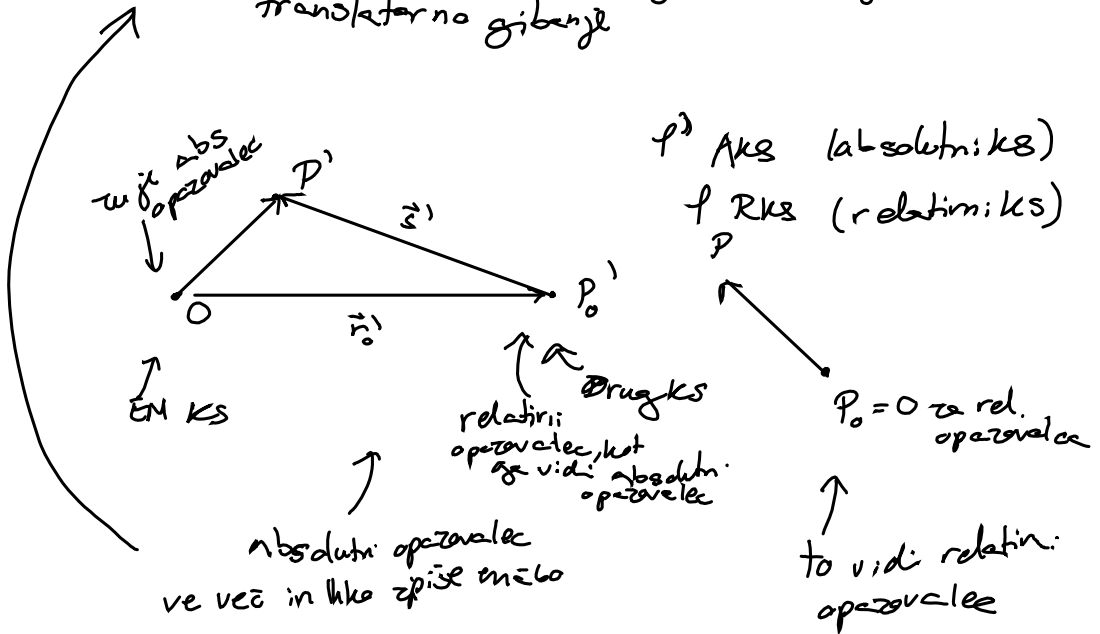
($P_0 \in E$, $P_0': \mathbb{R} \rightarrow E$, $U: \mathbb{R} \rightarrow SO(3)$) tako da velja

$$t = t')$$

← pri analizi je bil Q konstanten

$$P' = P_0'(t) + Q(t)(P - P_0)$$

translacijsko gibanje rotacijsko gibanje



Trditev: Rotacijski del gibanja je neodvisen od trajice

Dokaz: $\tilde{P}_0, \tilde{P}_0', \tilde{Q}$

$$P_0' + Q(P - P_0) = \tilde{P}_0' + \tilde{Q}(P - \tilde{P}_0) \quad \forall P$$

$$P = P_0 \Rightarrow P_0' = \tilde{P}_0' + Q(P_0 - \tilde{P}_0)$$

$$\tilde{P}_0' + \tilde{Q}(P - \tilde{P}_0) + Q(P - P_0) = \tilde{P}_0' + \tilde{Q}(P - \tilde{P}_0)$$

$$\tilde{P}_0' + \tilde{Q}(P_0 - \tilde{P}_0) + Q(P - P_0) = \tilde{P}_0' + \tilde{Q}(P - \tilde{P}_0) \Rightarrow$$

$$Q(P - P_0) = \tilde{Q}(P - P_0) - \tilde{Q}(P_0 - \tilde{P}_0) =$$

$$\tilde{Q}((P - \tilde{P}_0) - (P_0 - \tilde{P}_0)) = \tilde{Q}(P - P_0)$$

$$\Rightarrow Q = \tilde{Q}$$

$$P = P(t) \Rightarrow$$

$$\dot{P}_0' = \dot{\tilde{P}}_0' + \dot{Q}(P - P_0) + Q(\dot{P})$$

$\uparrow$ $\vec{v}_{rel} = \dot{P}$ relativna hitrost
 $\vec{v}_0'$ translacijska hitrost $\xrightarrow{\text{rotacijske hitrost}}$

$$Q^T Q = I = Q Q^T$$

$$(Q^T)' Q + Q^T \dot{Q} = 0$$

$$\dot{Q}^T Q + Q^T \dot{Q} = 0$$

$$W = Q^T \cdot \dot{Q}$$

$$W^T = \dot{Q}^T Q$$

$$W^T = -W \Rightarrow W \text{ je po\u0161evno simetri\u010dni tenzor}$$

Linearne preslike, $\rightarrow$ **Tenzor** (druga\u010da reda)

Trditev: $W = Q^T \ddot{Q}$ je poševno simetrični tenzor
 Trditev: Naj bo W poševno simetrični tenzor (p.s.t.)
 definiran na trirazsežnem vektorskem prostoru.
 Potem obstaja vektor $\vec{\omega}$ da je $W\vec{a} = \vec{\omega} \times \vec{a}$
 za vsi $\vec{a}$

Dokaz:

$$W\vec{p} = \lambda \vec{p}$$

$$\vec{p} \cdot W\vec{p} = \lambda |\vec{p}|^2$$

$$W^T \vec{p} \cdot \vec{p}$$

$$\parallel$$

$$- W\vec{p} \cdot \vec{p} =$$

$$\parallel$$

$$- \lambda |\vec{p}|^2 = \lambda |\vec{p}|^2 \Rightarrow \lambda = 0$$

$\{\vec{p}, \vec{q}, \vec{r}\}$ ONB

$$W\vec{p} = \vec{0}$$

$$W\vec{q} = \underbrace{(W\vec{q} \cdot \vec{p})}_{\parallel 0} \vec{p} + \underbrace{(W\vec{q} \cdot \vec{q})}_{\neq 0} \vec{q} + (W\vec{q} \cdot \vec{r}) \vec{r}$$

$$W\vec{a} = W((\vec{a} \cdot \vec{p})\vec{p} + (\vec{a} \cdot \vec{q})\vec{q} + (\vec{a} \cdot \vec{r})\vec{r}) = \vec{a} \cdot \vec{q} (W\vec{q} \cdot \vec{r}) \vec{r} +$$

$$\cancel{\vec{a} \cdot \vec{p} W\vec{p}} + \cancel{\vec{a} \cdot \vec{q} W\vec{q}} + \vec{a} \cdot \vec{r} (W\vec{r} \cdot \vec{q}) \vec{q}$$

$$\vec{p} \times \vec{q} = \vec{r} \quad \text{in} \quad \vec{r} \times \vec{p} = \vec{q} \quad \text{ker ONB}$$

$$= (W\vec{q} \cdot \vec{r}) [(\vec{a} \cdot \vec{q})(\vec{p} \times \vec{q}) - (\vec{a} \cdot \vec{r})(\vec{r} \times \vec{p})] =$$

$$= (W\vec{q} \cdot \vec{r}) \vec{p} \times [(\vec{a} \cdot \vec{q})\vec{q} + (\vec{a} \cdot \vec{r})\vec{r}] = (W\vec{q} \cdot \vec{r}) (\vec{p} \times \vec{a})$$

$$\underbrace{\quad}_{\vec{a}} \quad \uparrow$$

$$(\vec{a} \cdot \vec{p}) \vec{p} = 0$$

$$\vec{\omega} = (W\vec{q} \cdot \vec{r})$$

Vektorja $\vec{w}$ pravimo osni vektor p.s.t w

$$\vec{w} = \vec{w}(w)$$

$$e_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is a permutation of } 123 \\ -1 & \text{if } ijk \text{ is like} \\ 0 & \text{if } ijk \text{ is not a permutation} \end{cases}$$

$$\vec{a} \times \vec{b} = \sum_{i,j,k=1}^3 e_{ijk} a_j b_k \vec{e}_i = e_{ijk} a_j b_k \vec{e}_i$$

↑ pogost zapis brez vsote
sumacijske konjunktije

če se indeks ponovi, se sumira

$$\vec{a} = \sum a_i \vec{e}_i = a_i \vec{e}_i$$

$$\vec{a} \cdot \vec{b} = a_i b_i$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

$$\delta_{ii} = 1$$

$$W_{ij} = e_{ijk} w_k \quad \leftarrow \begin{matrix} \text{seštevamo} \\ \text{samo po } k \end{matrix}$$

$$W \vec{a} = W_{ij} a_j \vec{e}_i$$

$$w \times \vec{a} = e_{ijk} w_j a_k \vec{e}_i$$

$$W_{ij} a_j = e_{ijk} w_j a_k \quad \forall a$$

$$W_{ij} a_j = e_{ikj} w_k a_j \quad \leftarrow \text{if } ijk \text{ then } ijk$$

$$\Rightarrow W_{ij} = e_{ikj} w_k = -e_{ijk} w_k$$

$$e_{ijk} e_{mnk} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm} \quad \text{velja (p-ke ne bo)}$$

$$e_{ijk} e_{mjk} = \delta_{im} \delta_{jj} - \delta_{ij} \delta_{jm} = 3 \delta_{im} - \delta_{im} = 2 \delta_{im}$$

$$e_{ijk} e_{ijk} = 6$$

$$e_{ijp} \omega_{ij} = -e_{ijp} e_{ijk} \omega_k = -2 \delta_{pk} \omega_k = -2 \omega_p$$

$$\omega_p = -\frac{1}{2} e_{ijp} \omega_{ij}$$

Trilinear $A \in \mathcal{L}(U, V)$

$$A^*(\vec{a} \times \vec{b}) = (A\vec{a}) \times (A\vec{b})$$

$$A^* \text{ adjungierte } A\text{-je} \quad A^{-1} = \frac{1}{\det A} (A^*)^T$$

Dabei: $\det A \neq 0$ $\vec{a}, \vec{b}, \vec{c}$ lin. unabh.

$$\det A = \frac{[A\vec{a}, A\vec{b}, A\vec{c}]}{[\vec{a}, \vec{b}, \vec{c}]}$$

$$(A\vec{a} \times A\vec{b}) \cdot \vec{c} = (A\vec{a} \times A\vec{b}) \cdot (A A^{-1} \vec{c}) = [A\vec{a}, A\vec{b}, A A^{-1} \vec{c}] =$$

$$= \det A [\vec{a}, \vec{b}, A^{-1} \vec{c}] = \det A (\vec{a} \times \vec{b}) \cdot A^{-1} \vec{c} =$$

$$= (\det A) A^T (\vec{a} \times \vec{b}) \cdot \vec{c} \Rightarrow A\vec{a} \times A\vec{b} = \det(A) A^T (\vec{a} \times \vec{b})$$

$$A\vec{a} \times A\vec{b} = A^* (\vec{a} \times \vec{b})$$

$$\det A = 0 \Rightarrow \forall \varepsilon > 0. \exists A_\varepsilon. \det A_\varepsilon = 0. \|A - A_\varepsilon\| < \varepsilon$$

$$\lim_{\varepsilon \rightarrow 0} A_\varepsilon \vec{a} \times A_\varepsilon \vec{b} = A^* (\vec{a} \times \vec{b})$$

$$A_\varepsilon \rightarrow A \quad A_\varepsilon^* \xrightarrow{2} A^*$$

cc pc so lin. unabh.

$$[\vec{a}, \vec{b}, \vec{c}] = \det A [A\vec{a}, A\vec{b}, A\vec{c}] \text{ holds}$$

$$A = Q \in O(3)$$

$$Q^* = \det(Q) (Q^{-1})^T = \det Q Q$$

$$\parallel$$

$$(Q^T)^T = Q$$

$$Q \in SO(3) \Rightarrow Q(\vec{a} \times \vec{b}) = (Q\vec{a}) \times (Q\vec{b})$$

$$W = Q^T \dot{Q} \quad \text{osnovni vektorju p.s.t pravimo rotacijski vektor}$$

Vektorju $\vec{\omega}' = Q \vec{\omega}$ pa pravimo vektor kotne hitrosti:
rotacije Q

$$Q^T \dot{Q} \vec{a} = \omega \times \vec{a}$$

$$\dot{Q} \vec{a} = Q(\omega \times \vec{a}) = \vec{\omega}' \times Q \vec{a}$$

tradicij

Osnovni vektor p.s.t. $W = \dot{Q} Q^T$ je vektor kontinuirane hitrosti rotacije Q

$$\text{Dokaz: } \dot{Q} Q^T \vec{a} = \vec{\omega} \times \vec{a}$$

$$\dot{Q} \vec{a} = \vec{\omega}' \times Q \vec{a}$$

$$\dot{Q} Q^T \vec{a} = \vec{\omega}' \times Q(Q^T \vec{a}) = \vec{\omega} \times \vec{a}$$

$$\left. \begin{array}{l} \dot{Q} \vec{a} = \vec{\omega}' \times Q \vec{a} \\ \dot{Q} Q^T \vec{a} = \vec{\omega}' \times Q(Q^T \vec{a}) = \vec{\omega} \times \vec{a} \end{array} \right\} \Rightarrow \vec{\omega} = \vec{\omega}'$$

$Q \in SO(3)$

$W_1 W_2$ ni prave simetrični tenzor

$[W_1, W_2] = W_1 W_2 - W_2 W_1$ je pravno sim. tenz

$\nwarrow$ komutator

$(W, +, [\cdot])$ je algebra

$\uparrow$
množica p.s.t

$(V, +, \times)$

$\uparrow$
prostor 3D vektorej

je tudi algebra
(L: je algebra)

Trditev: $\omega([W_1, W_2]) = \omega(W_1) \times \omega(W_2)$

$\omega \mapsto W(\omega)$ je homomorfizem algebr

Dokaz $[W_1, W_2]\vec{a} = (W_1 W_2 - W_2 W_1)\vec{a} =$
 $\vec{w}_1 \times (\vec{w}_2 \times \vec{a}) - \vec{w}_2 \times (\vec{w}_1 \times \vec{a}) =$
 $= (\vec{w}_1 \vec{a}) \vec{w}_2 - \cancel{(\vec{w}_1 \vec{w}_2) \vec{a}} - ((\vec{w}_2 \vec{a}) \vec{w}_1 - \cancel{(\vec{w}_2 \vec{w}_1) \vec{a}}) =$
 $= (\vec{w}_1 \times \vec{w}_2) \times \vec{a}$

Izrek: Najbo W p.s.t z enostkim osnim vektorjem $\hat{e}$.

Potem je e^{vW} rotacija okrog osi $\hat{e}$ za kot v

Izrek: KS $\mathcal{P}(t, P)$ in $\mathcal{P}'(t, P')$ sta v istem razredu KS

$\Leftrightarrow P' = P_0' + Q(P - P_0)$ prelika vsake
 premočrtno gibanje s konstantno brzino
 v KS $\mathcal{P}$ z premočrtno s konstantno brzino
 v KS $\mathcal{P}'$

Dokaz

($\Rightarrow$) očitno

$$(\Leftarrow) \mathcal{P} = P_0 + \vec{v}t \mapsto \mathcal{P}' = \mathcal{P}'(t)$$

$$\vec{0} = \vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \vec{v}_{re}$$

$$\vec{0} = \vec{a}_0 + t \dot{\vec{\omega}} \times \vec{v} + t \vec{\omega} \times (\dot{\vec{\omega}} \times \vec{v}) + 2\vec{\omega} \times \vec{v} \quad / : \vec{v}$$

$$\vec{0} = \vec{a}_0 \vec{v} + t (\dot{\vec{\omega}} \times (\dot{\vec{\omega}} \times \vec{v})) \cdot \vec{v}$$

$$\vec{0} = \vec{a}_0 \vec{v} - t (\vec{v} \times \vec{\omega})(\vec{\omega} \times \vec{v}) = \vec{a}_0 \vec{v} - t \|\vec{v} \times \vec{\omega}\|_2$$

$$\vec{v} = \lambda \vec{v}_0$$

$$\downarrow$$

$$0 = \lambda \vec{a}_0 \vec{v}_0 - t \lambda^2 \|\vec{v}_0 \times \vec{\omega}\|_2 \quad \forall \lambda$$

$$\Rightarrow \vec{a}_0 \vec{v}_0 = 0 \quad \wedge \quad \vec{v}_0 \times \vec{\omega} = 0 \quad \forall \vec{v}_0$$

$$\vec{a}_0 = 0 \Rightarrow \vec{\omega} = \vec{0}$$



$\Leftrightarrow$
 Q je konstanten

$$\Rightarrow P_0(t) = P_0' + \vec{c}t$$

Posledica (Prvi Newtonov zakon)

$KS \text{ je } IKS \Leftrightarrow \forall$ prosta materialna točka se
giblje premočno s konstantno brzino

← mora obstajati
prosta materialna točka

Dokaz

$(\Rightarrow) \checkmark$

Prosta materialna točka - izolirana
od vpliva ostalih

$(\Leftarrow) \checkmark$

Lahko bi namesto proste reket:
masno težišče
(prosta točka mora obstajati)

Newtonova enačba v relativnem koordinatnem sistemu ^{RKS}

VKS
 $m\vec{a} = \vec{F}$

inercialna sila
 rotacijska
 centrifugalna sila

$$m\vec{a}_{re} = \vec{F} - m\vec{a}_0 - m\vec{\omega} \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) -$$

fiktivne sile

$$-2m\vec{\omega} \times \vec{v}_{re} = \vec{F}_{rel}$$

inercialna sila
 Coriolisova sila

$$m\vec{a}_{re} = \vec{F}_{re}$$

$$\vec{F} = Q^T \vec{F}' \quad F_{re} = Q^T F'_{rel}$$

Trditw: F' potencijalna $\Rightarrow F$ potencijalna

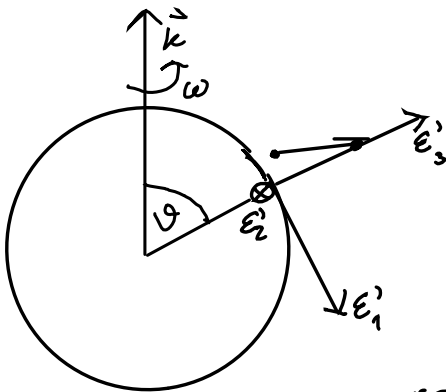
Dokaz: $\vec{F}' = -\text{grad}_{p'} U' = \left(\frac{\partial U'}{\partial p'} \right)^T = - \left(\frac{\partial U}{\partial p} \frac{\partial p}{\partial p'} \right)^T =$

$$U(p) = U'(p') = U'(p_0' + Q(p - p_0'))$$

$$p = Q^T(p' - p_0') + p_0$$

$$\Rightarrow -Q \left(\frac{\partial U}{\partial p} \right)^T = Q \vec{F} \Rightarrow \vec{F} = - \left(\frac{\partial U}{\partial p} \right)^T = -\text{grad}_p U$$

Foucaultovo nihalo



$m \vec{a}_{rel} = F_g + \vec{T} - m \vec{a}_o - m \vec{\omega} \times \vec{r} - m \vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m \vec{\omega} \times \vec{v}_{rel}$

(Annotations: $\vec{T}$ is labeled "sila vrvce", $\vec{\omega}$ is labeled "rotacija o konst.", $\vec{v}_{rel}$ is labeled "10⁻⁸")

$$\vec{\omega}' = \vec{\omega} k$$

$$\omega = \frac{2\pi}{24 \cdot 3600} \text{ /s} < 10^{-4}$$

$$\vec{a}_o = \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\vec{a} = \vec{a}_o + \vec{\omega} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2 \vec{\omega} \times \vec{v}_{rel} + \vec{a}_{rel}$$

(Annotations: Under $\vec{a}_o$ and $\vec{a}_{rel}$ are vertical arrows pointing to 0. Under $\vec{\omega} \times \vec{r}$ and $\vec{\omega} \times (\vec{\omega} \times \vec{r})$ are vertical arrows pointing to 0.)

pospešek KS nihala

gleda na KS, ki se vrti skupaj z zemljo

$$F_g - m \vec{a}_o = m \vec{g}$$

$$\vec{T} = T \cdot \frac{\vec{r}}{r} = \frac{L_1}{l} \vec{e}_1 + \frac{l - L_1}{l} \vec{e}_3 - \frac{L_2}{l} \vec{e}_2$$

itd. neli

Sistem materialnih točk

$$P_1, \dots, P_N$$

$$m_i \ddot{\vec{P}}_i = \vec{F}_i + \sum_{\substack{j=1 \\ j \neq i}}^N \vec{F}_{ji} \quad \vec{F}_{ji} = -\vec{F}_{ij}$$

$$\sum_{i=1}^N m_i \vec{P}_i = \sum_{i=1}^N \vec{F}_i + \sum_{i \neq j}^N \vec{F}_{ji}$$

1) 0

$$\vec{P}_* = 0 + \sum_{i=1}^N m_i (\vec{P}_i - 0)$$

$$m = \sum_{i=1}^N m_i$$

$$\vec{P}_*^0 = \frac{1}{m} \sum_{i=1}^N m_i \vec{P}_i^0 \quad \vec{P}_*^{\infty} = \frac{1}{m} \sum_{i=1}^N m_i \vec{P}_i^{\infty}$$

$$m \ddot{\vec{P}}_* = \vec{F} = \sum_{i=1}^N m_i \ddot{\vec{P}}_i \quad \text{Enačba gibanja masnega središča}$$

