

$$x = 0, 1$$

Pokaži da vrijedi *

$$x = \sum_{i=1}^{\infty} (2^{-4i} + 2^{-4i-1})$$

b) binarni zapis za x

c) zapis u IEEE formi

IEEE754

binarna notacija $P(2, 24, -125, 128)$

$$(-1)^{\sigma} (1+m) 2^{\tilde{e}-127}$$

m dulžine 23

e dulžine 8

σ dulžine 1

binarna notacija $P(2, 53, -1021, 1024)$

$$(-1)^{\sigma} (1+m) 2^{\tilde{e}-1023}$$

m 52

e 11

σ 1

$$a) X = \sum_{i=1}^{\infty} 2^{-4i} = \frac{\frac{1}{16}}{1 - \frac{1}{16}} + \frac{1}{2} \frac{\frac{1}{16}}{1 - \frac{1}{16}} = \frac{2}{2 \cdot 16} \frac{16}{16 \cdot 5} = \frac{1}{10} = 0,1$$

$$b) 0,0001100110011 = 0,0\overline{0011}_{(2)}$$

$$c) \begin{aligned} &1,1\overline{0011} \cdot 2^{-4} = \\ &1 + 0,1\overline{0011} \cdot 2^{-4} \end{aligned}$$

$$0,100110\dots\dots 001101$$

$$\tilde{e} - 127 = -4 \Rightarrow e = 123 = 1111011$$

123 : 2 = 61	1
61 : 2 = 30	1
30 : 2 = 15	0
15 : 2 = 7	1
7 : 2 = 3	1
3 : 2 = 1	1
1 : 2 = 0	1

$$x = 2^{-1} + 2^{-k} + 2^{-t}$$

$$y = 2^{-1} + 2^{-k}$$

$$k = \frac{t}{2} + 1$$

$$t = 2k - 2$$

$x^2 + y^2$ izračunamo z izračun

$$x \cdot x - y \cdot y$$

z obravnavo relativne napake pletilk de izračunani dovolj stabilni

$$x^2 = (2^{-1} + 2^{-k} + 2^{-t})^2 = 2^{-2} + 2^{-2k} + 2^{-2t} + 2 \cdot 2^{-1-k} + 2 \cdot 2^{-k-t} + 2 \cdot 2^{-t-1} =$$

$$\cancel{2^{-2}} + \cancel{2^{-2k}} + 2^{-4k+4} + \cancel{2^{-2t}} + \cancel{2^{-k}} + 2^{-k-t+1} + 2^{-t}$$

$$y^2 = (2^{-1} + 2^{-k})^2 = 2^{-2} + \cancel{2^{-1-k}} + \cancel{2^{-k}} + 2^{-2k}$$

$$x^2 - y^2 = 2^{-4k+4} + 2^{-k-t+1} + 2^{-t}$$

$$= 2^{-2t} + 2^{-t} + 2^{-k-t+1}$$

$$fl(x) = 0,01 \dots 1 \dots 101 \dots 1 \dots =$$

$$= 0,01 \dots 1 \dots 11$$

$$fl(y) = 0,01 \dots 1 \dots 001 \dots = 0,01 \dots 1 \dots 010$$

nivelo denar zad:

zato izberemo številke
soda zadnja številka, ker jo
1 ne moremo -2^k zadrževati
v ~~zadrževati~~ zadrževati

Y

22.10

$$g(x) = -x^2 + 8x - 12 \quad x_{r+1} = g(x_r)$$

$$\lim_{r \rightarrow \infty} x_r = 4 \quad \text{za} \quad \forall x_0 \in (3, 5)$$

 $\Leftrightarrow$

$$\lim_{r \rightarrow \infty} |x_r - 4| = 0$$

$$= |x_{r+1} - 4|$$

$$\lim_{r \rightarrow \infty} |x_{r+1} - 4| = \lim_{r \rightarrow \infty} |-x_r^2 + 8x_r - 12| =$$

$$= \lim_{r \rightarrow \infty} |x_r - 4|^2 = \lim_{r \rightarrow \infty} |x_{r-1} - 4|^4 = \dots = \lim_{r \rightarrow \infty} |x_0 - 4|^{2^{r+1}}$$

konverguje za $|x_0 - 4| < 1 \rightarrow x_0 \in (3, 5)$

prosto

Red konvergence

α negibna točka

$$g'(\alpha) = g''(\alpha) = \dots = g^{(r-1)}(\alpha) = 0 \quad g^{(r)}(\alpha) \neq 0$$

potem je **red konvergence** enak p

- a) pokažite da lahko $\sqrt{\quad}$ pozitivnega števila a izračunamo z iteracijo
$$x_{r+1} = x_r \cdot \frac{x_r^2 + 3a}{3x_r^2 + a}$$
- b) delajte red konvergence
- c) dokažite da iteracija konvergira k $\sqrt{a}$ za $\forall$ začetni približek $x_0 > 0$

1. Preverimo če je $\sqrt{a}$ negibna točka iteracijske funkcije

$$\sqrt{a} \cdot \frac{\sqrt{a}^2 + 3a}{3\sqrt{a}^2 + a} = \sqrt{a} \cdot \frac{4a}{4a} = \sqrt{a}$$

1. negibnost
2. privlačnost

2. preverjamo privlačnost $|g'(\sqrt{a})| < 1$

$$g(x) = \frac{x^3 + 3ax}{3x^2 + a}$$

$$g'(x) = \frac{(3x^2 + 3a)(3x^2 + a) - 6x(x^3 + 3ax)}{(3x^2 + a)^2}$$

$$g'(\sqrt{a}) = \frac{(3a + 3a)(3a + a) - 6a^2 - 18a^2}{(3a + a)^2}$$

$$= 0 < 1$$

$$b) g''(x) = \left(3 \frac{(x^2 + a)(3x^2 + a) - 6x^2(x^2 + 3a)}{(3x^2 + a)^2} \right)'$$

$$= 3 \frac{(2x + a)(3x^2 + a) + 6x(x^2 + a) - 4x(x^2 + 3a) - 4x^3}{(3x^2 + a)^3}$$

$$= 3 \frac{(2x^3 + 2ax^2 + 3ax^2 + a^2 + 6x^3 + 6ax - 4x^3 - 12ax - 4x^3)}{(3x^2 + a)^3}$$

$$= -3 \frac{(6x^3 + 3ax^2 + 2ax + a + 6x^3 + 6a - 4x^3 + 12xa - 4x^3)}{(3x^2 + a)^3}$$

$$g'' = \frac{h_1}{h_2} = \frac{48xa(x^2 + a)}{(3x^2 + a)^3}$$

$$g''' = h_1' h_2 + h_1 h_2' = h_1'(x^2 + a) + h_1(x) 2x$$

$$g'''(\sqrt{a}) = h_1'(\sqrt{a}) \cdot 0 + h_1(\sqrt{a}) 2\sqrt{a}$$

$$h_1 \sqrt{a} = \frac{48a\sqrt{a}}{3a + a}$$

$$g''' \sqrt{a} = \frac{48 \cdot 2a^2}{(4a)^3} = \frac{3}{2a} > 0 \neq 0$$

$\Rightarrow$ red konvergence $= 3$

$$c) \quad g(x) = x \frac{x^2 + 3a}{3x^2 + a}$$

ločimo primeri ko $x_0 < \sqrt{a}$

$$x_1 > x_0?$$

$$x_1 > \sqrt{a} \text{ ali } x_1 < \sqrt{a}?$$

$$x_1 = \cancel{x_0} \frac{x_0^2 + 3a}{3x_0^2 + a} \cancel{> x_0}$$

$$x_0^2 + 3a > 3x_0^2 + a$$

$$2a > 2x_0^2 \quad \checkmark$$

$$\sqrt{a} > x_0 \quad \checkmark$$

$$\underline{\underline{x_1 < \sqrt{a}}}$$

$$x_0 (x_0^2 + 3a) < \sqrt{a} (3x_0^2 + a)$$

$$\Leftrightarrow$$

$$\sqrt{a} (x_0^2 + 3a) < \sqrt{a} (3x_0^2 + a)$$

$$2a < 2x_0^2$$

$$\text{sledi } x_1 < \sqrt{a}$$

torej x_i narašča in so omejen
konvergenca nekon

konvergenca k $\sqrt{a}$?

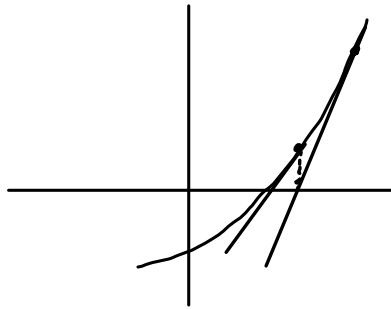
$$x_0 \in (\sqrt{a}, \infty)$$

pokažemo da imamo padejočo
nevezdno omejeno zaporedje

Tangentna metoda

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$x_{r+1} = g(x_r)$$



Babilonska metoda za računanje $\sqrt{a}$ $a > 0$
temelji na iteraciji

$$x_{r+1} = \frac{1}{2} \left(x_r + \frac{a}{x_r} \right)$$

a) Preverite, da iteracija ustreza tangentni metodi
za funkcijo $f(x) = x^2 - a$

b) določite red konvergence

c) Dokazite da iteracija konvergira k $\sqrt{a}$ za $\forall x_0 > 0$

$$\begin{aligned} \text{a)} \quad g(x) &= \frac{1}{2} \left(x + \frac{a}{x} \right) \stackrel{?}{=} x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - a}{2x} = \\ &= \frac{2x^2 - x^2 + a}{2x} = \frac{1}{2} \left(x + \frac{a}{x} \right) \quad \checkmark \end{aligned}$$

$$\text{b)} \quad g'(x) = \frac{1}{2} \left(1 - \frac{a}{x^2} \right) \rightarrow g'(\sqrt{a}) = 0$$

$$g''(x) = \frac{1}{2} \left(\frac{a}{x^3} \right) \stackrel{?}{=} \frac{a}{2x^3} \quad \frac{1}{\sqrt{a}} > 0$$

red konvergence je 2

$$c) \quad g(x) = \frac{1}{2} \left(x + \frac{a}{x} \right)$$

$$g \text{ nevera } \Rightarrow \text{ } x \in (\sqrt{a}, \infty)$$

$$\sqrt{a} < x_{r+1} < x_r$$

$$\bar{s} \in \mathbb{Z}$$

$$\sqrt{a} < \frac{1}{2} \left(x_r + \frac{a}{x_r} \right) < x_r$$

$$2\sqrt{a}x_r < x_r^2 + a < 2x_r^2$$

$$\underbrace{\hspace{10em}}_{a < x_r^2} \checkmark$$

$$x_r^2 - 2\sqrt{a}x_r + a > 0$$

$$(x_r - \sqrt{a})^2 > 0$$

pede jogar o novo valor amejorado novamente

$$x_0 \in (0, \sqrt{a}) \Rightarrow \underline{\underline{x_1 \in (\sqrt{a}, \infty)}}$$

$$x_1 = \frac{1}{2} \left(x_0 + \frac{a}{x_0} \right) > \sqrt{a}$$

$$x_0^2 + a > 2x_0\sqrt{a}$$

$$x_0^2 - 2x_0\sqrt{a} + a > 0$$

$$(x_0 - \sqrt{a})^2 > 0 \quad \checkmark$$

Naj bo $f \in \mathcal{C}^2$ a njena enostavna ničla

a) Dokazite, da metoda

$$x_{r+1} = x_r - \frac{2f(x_r) \cdot f'(x_r)}{2f'(x_r)^2 - f(x_r)f''(x_r)} \quad \text{ustreza tangentni}$$

$$\text{metodi za funkcijo } F(x) = \frac{f(x)}{\sqrt{|f'(x)|}}$$

b) poenostavite metodo za $f(x) = x^2 - a$

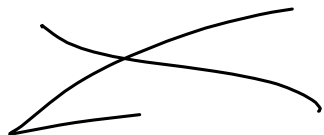
$$a) \quad F'(x) = \frac{f'(x)\sqrt{|f'(x)|} - \sqrt{|f'(x)|}}{|f'(x)|} =$$

$$\frac{f''(x)}{2\sqrt{|f'(x)|}} \cdot \frac{f'(x)}{|f'(x)|}$$

$$= \frac{\frac{f''(x)}{\sqrt{|f'(x)|}} \left(1 - \frac{f''(x)}{2|f'(x)|}\right)}{|f'(x)|} = \frac{\operatorname{sgn} f \left(1 - \frac{1}{2} \operatorname{sgn} f\right)}{\sqrt{|f'(x)|}} =$$

$$= \frac{\operatorname{sgn} f - \frac{1}{2}}{\sqrt{|f'(x)|}}$$

$$\frac{F(x)}{F'(x)} = \frac{f(x) \cancel{\sqrt{|f'(x)|}}}{\cancel{\sqrt{|f'(x)|}} \left(\operatorname{sgn} f - \frac{1}{2}\right)} = \frac{f(x)}{\operatorname{sgn} f - \frac{1}{2}}$$



1.1

$$P(2, 3, -1, 3) \quad \pm m \cdot b^e \quad L \leq e \leq U$$

$$P(b, t, L, U)$$

base		e: -1	1	2	3
0					
0,100		0,01	1	10	100
0,101		0,0101	1,01	10,1	101
0,110		0,011	1,1	11	110
0,111		0,0111	1,11	11,1	111

1.2

$$P(2, 9, -10, 10)$$

$$X = 47,712$$

$$\begin{aligned} 47 : 2 &= 23 \text{ in } 1 \\ 23 : 2 &= 11 \text{ in } 1 \\ 11 : 2 &= 5 \text{ in } 1 \\ 5 : 2 &= 2 \text{ in } 1 \\ 2 : 2 &= 1 \text{ in } 0 \\ 1 : 2 &= 0 \text{ in } 1 \end{aligned}$$

$$1 + 2 + 2^2 + 2^3 + 2^5 =$$

$$\begin{array}{r} 32 \\ 8 \\ 4 \\ 3 \\ \hline 47 \end{array} \quad 10111$$

$$\begin{array}{r} 0,712 \cdot 2 \\ \hline 1,424 \end{array} \quad 1$$

$$0,101111010 \cdot 10^6$$

$$\begin{array}{r} 0,424 \cdot 2 \\ \hline 0,848 \end{array} \quad 0$$

$$\begin{array}{r} 0,848 \cdot 2 \\ \hline 1,696 \end{array} = 1$$

$$\begin{array}{r} 0,696 \cdot 2 \\ \hline 1,392 \end{array} \quad 1$$

$$\begin{array}{r} 0,392 \cdot 2 \\ \hline 0,784 \end{array} \quad 0$$

$$0,1 = \sum_{i=1}^{\infty} (2^{-4i} + 2^{-4i-1})$$

$$\sum_{i=1}^{\infty} (2^{-4})^i = \frac{1}{1-2^{-4}} = \frac{2^4}{2^4-1}$$

$$\frac{1}{2} \sum_{i=1}^{\infty} (2^{-4})^i = \frac{2^3}{2^4-1}$$

$$\frac{\quad}{+} = \frac{2^4+2^3}{15} = \frac{24}{15} = \frac{8}{5} = \frac{16}{10} = 1,6$$

$$\frac{3}{2} \frac{16}{15} = \frac{8}{5} =$$

1 2.3

$$f(x) = x^5 - 10x + 1$$

$$f(0) = 1$$

$$f(0,2) = 0,2^5 - 2 + 1 < 0$$

ma vsej eno ničlo

oz. ima $\exists \epsilon$, ma skrajno c. t. d.

$$f'(x) = 5x^4 - 10 = 0$$

$$x^4 = 2$$

$$x = \sqrt[4]{2} \text{ ni med } 0 \text{ in } 0,2$$

b)

$$x_0 = 0$$

$$x_{r+1} = g(x_r)$$

$$g(x) = (x^5 + 1)/10$$

$$x_{r+1} = \frac{(x_r^5 + 1)}{10}$$

$$\lim_{r \rightarrow \infty} f(x_r)$$

$$g'(x) = \frac{5x^4}{10}$$

$$g'(0) = 0 <$$

$$g(g(0)) = g\left(\frac{1}{10}\right) = \frac{\frac{1}{10^5} + 1}{10} = \frac{10^5 + 1}{10^6}$$

$$g(x) = -x^2 + 8x - 12$$

$$g(x) = x$$

$$-x^2 + 8x - 12 - x = 0$$

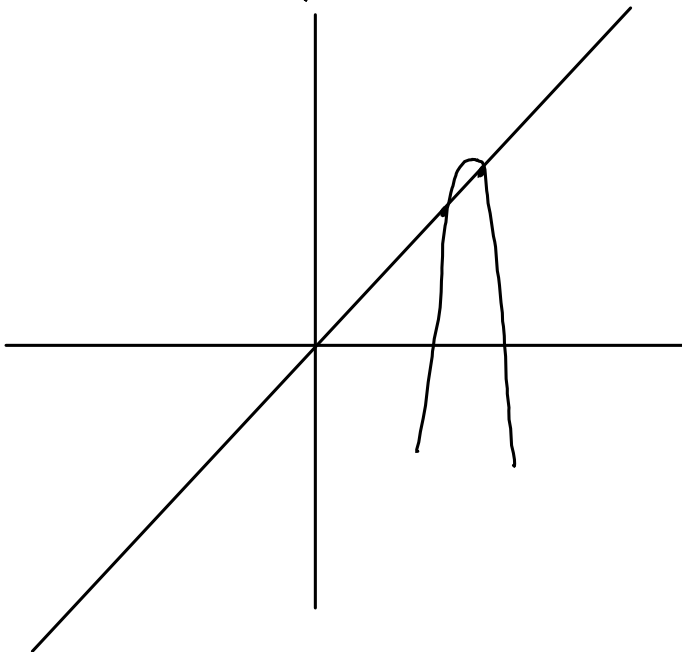
$$-x^2 + 7x - 12 = 0$$

$$-(x-3)(x-4) = 0$$

$$g'(x) = -2x + 8 \quad |-2x + 8| < 1$$

$$g'(3) = 2 \quad \text{addegr} \quad \in \left(\frac{7}{2}, \frac{9}{2}\right)$$

$$g'(4) = 0 \quad \text{prilozor}$$



$$b_n = \frac{10}{3}b_{n-1} - b_{n-2}$$

$$b_{n-2} = \frac{10}{3}b_{n-1} - b_n$$

2,7 naloga

$$f(x) = (x^2 - a)x = x^3 - ax$$

$$g(x) = x \frac{x^2 + 3a}{3x^2 + a} \quad \text{je}$$

negibne točke x : $\sqrt{a}, -\sqrt{a}, 0$

za $x_0 > 0$ konvergira k $\sqrt{a}$

.....

$$x_r - \sqrt{a} = g(x_{r-1}) - \sqrt{a} = \frac{(x_{r-1} - \sqrt{a})^3}{3x_{r-1}^2 + a} < 0$$

$$x_0 - \sqrt{a} < 0$$

$$\underbrace{\quad}_{>0} \Rightarrow x_{r-1} - \sqrt{a} < 0$$

ali je strogo naraščajoča? $x_r < \sqrt{a}$

2.19 naloga

$$f(x) = x^2 - a \quad a > 0$$

Hallyjeva metoda

$$g(x) = x - \frac{2f(x)f'(x)}{2f'(x)^2 - f(x)f''(x)}$$

$$f'(x) = 2x \quad f''(x) = 2$$

$$g(x) = x - \frac{2(x^2 - a)2x}{2(2x)^2 - (x^2 - a)2} =$$

$$= x - \frac{4x^3 - 4ax}{8x^2 - 2x^2 + 2a} = \frac{2x^3 + 6ax}{6x^2 + 2a} =$$

$$= \frac{x(x^2 + 3a)}{3x^2 + a}$$

2.15 nelage

$$x_{r+1} = \frac{x_{r-1}x_r + a}{x_{r-1} + x_r}$$

$$x_0, x_1 > \sqrt{a}$$

sekantna metoda

$$x_{r+1} = x_r - \frac{f(x_r)(x_r - x_{r-1})}{f(x_r) - f(x_{r-1})}$$

$$f(x) = x^2 - a$$

$$\begin{aligned} x_{r+1} &= x_r - \frac{(x_r^2 - a)(x_r - x_{r-1})}{x_r^2 - x_{r-1}^2} = \\ &= \frac{\cancel{x_r^3} - x_{r-1}^2 x_r - \cancel{x_r^3} + a x_r + x_{r-1} x_r^2 - a x_{r-1}}{x_r^2 - x_{r-1}^2} = \\ &= \frac{x_{r-1} x_r (x_r - x_{r-1}) + a(x_r - x_{r-1})}{x_r^2 - x_{r-1}^2} = \frac{x_{r-1} x_r + a}{x_r + x_{r-1}} \end{aligned}$$

$$\underline{x_{r+1} < x_r}$$

$$\begin{aligned} x_{r+1} &< \frac{x_{r-1}x_r + a}{x_r + x_{r-1}} \Rightarrow x_{r+1}x_r + x_{r+1}x_{r-1} < x_{r-1}x_r + a \\ &< \\ 2x_{r+1}x_r &< \cancel{x_{r-1}x_r} + \cancel{a} \\ &\quad \quad \quad \downarrow a < x_r \\ x_{r+1} &< \frac{x_r}{2} < x_r \quad \dots \end{aligned}$$

3.3 norma

$$N_{\infty}(A) = \max |a_{ij}|$$

1) N ni matrica norma

$$N(A)N(B) = N(AB)$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \quad \times$$

2) $\|A\| = n N_{\infty}(A)$ je matrica norma
pozitivna definitna

• $\|A\| \geq 0$: sledi iz dds, in $n > 0$

$$\|A\| = 0 \Leftrightarrow n = 0 \vee a_{ij} = 0 \forall a_{ij} \Rightarrow A = 0$$

• homogenost

$$\|\alpha A\| = n N(\alpha A) = n |\alpha| N(A) = |\alpha| \|A\| \quad \checkmark$$

• trik. neenakost

$$\|A+B\| = n N_{\infty}(A+B) = n \max_{ij} |a_{ij} + b_{ij}| \leq n \max_{ij} |a_{ij}| + n \max_{ij} |b_{ij}|$$

• multiplikativnost

$$\leq \|A\| + \|B\|$$

$$\underline{\underline{\|A \cdot B\| \leq \|A\| \|B\|}}$$

$$n \left\| \sum_{c=1}^n a_{ic} b_{cj} \right\| \leq n \sum_{c=1}^n |a_{ic}| |b_{cj}|$$

$$n \max |a_{ij}| \left| \sum_{c=1}^n b_{cj} \right| \leq n \max |a_{ij}| n \max |b_{cj}| \leq \|A\| \|B\|$$

3.8 relage

$$\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |k_{ij}|^2}$$

$$\|A\|_2 = \max \sqrt{\lambda(A^H A)}$$

laste vektorene av $A^H A$

$$1) \frac{1}{\sqrt{n}} \|A_F\| \leq \|A\|_2 \leq \|A\|_F$$

$$\det(A^H A - \lambda I) = 0$$

3.10

$$\|A\|_2 \leq \|A\|_2$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^H = A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$A^H A = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix}$$

Vespera Φ

LU razcep

• brez pivotiranja (zadnjič :))

• delno pivotiranje $PA=LU$

$$Ax=b$$

$$LUx = Pb$$

$$1) Ly = Pb$$

$$2) Ux = y$$

• kompletno pivotiranje $Ly = Pb$ 2) $Uz = y$ 3) $Q^{-1}x = z$

$$A = \begin{bmatrix} 2 & 1 & -2 & 1 \\ 2 & 1 & -4 & 2 \\ 3 & -2 & 3 & -1 \\ -1 & 3 & -1 & 1 \end{bmatrix}$$

izračunaj LU razcep in kolo $\det(A)$

$$\begin{array}{c} \text{3} \\ \curvearrowright \end{array} \begin{bmatrix} 3 & -2 & 3 & -1 \\ 2 & 1 & -4 & 2 \\ 2 & 1 & -2 & 1 \\ -1 & 3 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & -2 & 3 & -1 \\ \frac{2}{3} & \frac{7}{3} & -6 & \frac{8}{3} \\ \frac{2}{3} & \frac{7}{3} & -4 & \frac{5}{3} \\ -\frac{1}{3} & \frac{7}{3} & 0 & \frac{2}{3} \end{bmatrix}$$

$$1 - (2)\left(\frac{2}{3}\right)$$

$$P \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

$$1 - (-2)\left(\frac{2}{3}\right)$$

$$\begin{bmatrix} 3 & -2 & 3 & -1 \\ \frac{2}{3} & \frac{7}{3} & -6 & \frac{8}{3} \\ \frac{2}{3} & 1 & 2 & -1 \\ -\frac{1}{3} & 1 & 6 & -2 \end{bmatrix} \rightsquigarrow$$

$$\rightsquigarrow \begin{bmatrix} -3 & -2 & 3 & -1 \\ \frac{2}{3} & \frac{7}{3} & -6 & \frac{8}{3} \\ -\frac{1}{3} & 1 & 6 & -2 \\ \frac{2}{3} & 1 & 2 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -3 & -2 & 3 & -1 \\ \frac{2}{3} & \frac{7}{3} & -6 & \frac{8}{3} \\ -\frac{1}{3} & 1 & 6 & -2 \\ \frac{2}{3} & 1 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$$-1 - (-\frac{2}{3})$$

$$U = \begin{bmatrix} -3 & -2 & 3 & -1 \\ & \frac{7}{3} & -6 & \frac{8}{3} \\ & & 6 & -2 \\ & & & -\frac{1}{3} \end{bmatrix} \quad L = \begin{bmatrix} 1 & & & \\ \frac{2}{3} & 1 & & \\ -\frac{1}{3} & 1 & 1 & \\ \frac{2}{3} & 1 & \frac{1}{3} & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\det(A) = \frac{\det(U) \det(L)}{\det(P)} = \det(U) = (+7)(-2) = -14$$

$$\parallel \pm 1 = (-1)^{\text{št. am. vrstic}}$$

$$\parallel 2$$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 2 & -6 \\ -3 & 3 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 8 \\ 14 \\ 0 \end{bmatrix}$$

LU s kompletnim pivotingom + sistem $Ax=b$

$$A = \begin{bmatrix} 4 & 2 & -6 \\ 1 & 2 & -3 \\ -3 & 3 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -6 & 2 & 4 \\ -3 & 2 & 1 \\ 3 & 3 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -6 & 2 & 4 \\ \frac{1}{2} & 1 & -1 \\ \frac{1}{2} & 4 & -1 \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} -6 & 2 & 4 \\ \frac{1}{2} & 4 & -1 \\ \frac{1}{2} & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -6 & 2 & 4 \\ \frac{1}{2} & 4 & -1 \\ \frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & & \\ \frac{1}{2} & 1 & \\ \frac{1}{2} & \frac{1}{4} & 1 \end{bmatrix} \quad U = \begin{bmatrix} -6 & 2 & 4 \\ & 4 & -1 \\ & & -\frac{3}{4} \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} & 1 & 1 \\ 1 & & \end{bmatrix}$$

zamjenjuje
stoljce

$$Ly = Pb$$

$$\begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ \frac{1}{2} & \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 8 \end{bmatrix}$$

$$y_1 = 14$$

$$-\frac{14}{2} + y_2 = 0 \rightsquigarrow y_2 = +7$$

$$y_3 = -\frac{3}{4}$$

$$Uz = y$$

$$x = Qz$$

a) $A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & 5 & 2 & 0 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ izračunajte LU razcep brez pivotiranja
kaj opazite

b) Naj bo $\begin{bmatrix} a_{11} & b_1 & & \\ c_1 & a_{22} & & \\ & \ddots & \ddots & \\ & & c_{n-1} & a_{nn} & b_n \end{bmatrix}$ splošna 3-diagonalna matrika

zapišite algoritem za razcep te tridiagonalne matrike in preštejte št. operacij
koliko operacij potrebujemo za reševanje sistema

$$AX = Z$$

a) $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & 7 & 2 & 0 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & 7 & 2 & 0 \\ 0 & -\frac{2}{3} & \frac{11}{3} & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & 7 & 2 & 0 \\ 0 & -\frac{2}{3} & \frac{11}{3} & 6 \\ 0 & 0 & \frac{7}{11} & -\frac{20}{11} \end{bmatrix}$ $6 \cdot 7 = 42$
 $2 - \frac{42}{11} = \frac{-20}{11}$
 $L = I$ $U = A_{uz}$

b) for i in range $(1, n)$:
 $L[i][i-1] = A[i][i-1] / A[i-1][i-1]$
 $U[i][i] = A[i][i] - L[i][i-1] \cdot A[i-1][i]$

$$u_1 = a_1$$

for $i = (1:n)$

$$l_i = \frac{c_i}{u_i}$$

$$u_{i+1} = a_{i+1} - l_i \cdot b_i$$

end

$n-1$ delitev za l

$n-1$ množenj

$n-1$ odštevanj

$3 \cdot (n-1)$ operacij

$$AX = Z$$

$$LY = Z: \quad y_1 = z_1 \quad y_2 = l_1 y_1 + z_2 \quad \dots \quad y_i = z_i - l_{i-1} y_{i-1}$$

$2 \cdot (n-1)$ operacij

$$UX = Z: \quad x_n = u_n^{-1} z_n$$

$$\cancel{x_{n-1}} \quad b_{n-1} x_n + u_{n-1} x_{n-1} = z_n$$

$$x_{n-1} = \frac{z_n - b_{n-1} x_n}{u_{n-1}} \quad x_i = \frac{z_{i+1} - b_i x_{i+1}}{u_i}$$

$$= 3 \cdot (n-1) + 1 = 3n-2 \text{ operacij}$$

skupen $5n-4$ operacij

Opisite postopek za reševanje sistema linearnih enačb

$$\begin{bmatrix} U & -I \\ B & L \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \text{ kjer je } B=LU \text{ nesingularna}$$

Prešteje število operacij

$$\begin{bmatrix} Ux - y \\ Bx + Ly \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$Bx + Ly = L Ux + Ly = L(Ux + Iy)$$

$$y = Ux - a \quad Ux = y + a$$

$$\begin{aligned} Bx + Ly &= L(Ux + Ux - a) = L(2Ux - a) = \\ &= L(2y + a) = b \end{aligned}$$

1. $Lz = b$ n^2 operacij

2. $y = \frac{z - a}{2}$ $2n$ operacij

3. $Ux = a + y$ $n + n^2 + n =$
obrati
suma

skupaj: $2n + 4n$

15.11

Y menjela
eno uo

Razcep Choleskega

$A \in \mathbb{R}^{n \times n}$ simetrična, pozitivno definitna
($\forall x \in \mathbb{R}^n - \{0\}, x^T A x > 0$)

$$A = V \cdot V^T \quad V = \begin{array}{c} \text{nesingularna s} \\ \text{pozitivnimi diag. elementi} \end{array}$$

for $j = 1:n$

$$V_{j,j} = \sqrt{a_{j,j} - \sum_{k=1}^{j-1} V_{j,k}^2}$$

for $j = i+1:n$

$$V_{j,i} = \frac{1}{V_{i,i}} \left(a_{j,i} - \sum_{k=1}^{j-1} V_{j,k} V_{i,k} \right)$$

a) Dobroš testar Choleskega (v)

$$A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 8 & -2 & 8 \\ -2 & -2 & 14 & -11 \\ 3 & 8 & -11 & 15 \end{bmatrix}$$

b) Naj bo

$$A = \begin{bmatrix} 4 & 6 & 2 & -4 \\ 6 & 18 & 0 & 3 \\ 2 & 0 & 3 & -4 \\ -4 & 3 & -4 & \alpha \end{bmatrix}$$

i) za kakšno α je to poz. def.

ii) ~~$\alpha = 23$~~ $\alpha = 23$ razlikujejo

$$Ax = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 1 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & & & \\ 2 & 2 & & \\ -2 & 1 & 3 & \\ 3 & 1 & -2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 6 & 2 & -4 \\ 6 & 18 & 0 & 3 \\ 2 & 0 & 3 & -4 \\ -4 & 3 & -4 & \alpha \end{bmatrix}$$

$$V = \begin{bmatrix} 2 & & & \\ 3 & 3 & & \\ 1 & -1 & 1 & \\ -2 & 3 & 1 & \sqrt{\alpha - 14} \end{bmatrix}$$

$$V_{ji} = \frac{1}{V_{ii}} \left(a_{ji} - \sum_{k=1}^{j-1} V_{jk} V_{ik} \right)$$

$$V_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} V_{ik}^2}$$

$$V_{21} = \frac{1}{1} (2 - 0)$$

$$V_{22} = \sqrt{18 - 2^2} = 4$$

$$V_{32} = \frac{1}{2} (-2 - (-4)) = 1$$

$$V_{42} = \frac{1}{2} (8 - (3 \cdot 2)) = 1$$

$$\alpha > 14$$

26.11

Dani sta metriki A in $B \in \mathbb{R}^{n \times n}$, B je pozitivno definitna. sestavite učinkovit postopek za izračun sledi od $A^T B^{-1} A$ in preštejetе št. operacij

Ne kbi;

Reševanje sistemov nelinearnih enačb

$$\begin{aligned} f_1(x_1, \dots, x_n) &= 0 \\ f_2(x_1, \dots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, \dots, x_n) &= 0 \end{aligned} \quad \leadsto F(x_1, \dots, x_n) = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{bmatrix}$$

$$F(\underline{x}) = 0$$

Jakobijeva iteracija

$$G(x) = \begin{bmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{bmatrix} \quad x^{(r+1)} = G(x^{(r)})$$

Seidlove iteracije

$$x_i^{(r+1)} = g_i(x_1^{(r+1)}, \dots, x_{i-1}^{(r+1)}, x_i^{(r)}, \dots, x_n^{(r)})$$

- α je privlačna negibna točka, če obstaja kakšna metrična norma, da velja $\|J_G(\alpha)\| < 1$
(G mora biti odvedljiva v α)

Potem $\exists$ okolica $\Omega \ni \alpha$. $\forall \underline{x}^{(0)} \in \Omega$ zaporedje konvergira k α

Dan je sistem enačb

$$\begin{aligned}x &= \sin\left(\frac{2x-y}{4}\right) \\ y &= \cos\left(\frac{x+2y}{4}\right)\end{aligned}$$

- a) pri začetnem približku $x^0 = \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$ naredite dva koraka Jakobijeve in se idolno iteracije
- b) Pokazite, da iteraciji konvergira za vsak začetni približek

$$G(x, y) = \begin{bmatrix} \sin\left(\frac{2x-y}{4}\right) \\ \cos\left(\frac{x+2y}{4}\right) \end{bmatrix}$$

$$x^1 = G\left(\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sin\left(\frac{4\pi}{4}\right) \\ \cos\left(\frac{2\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x^2 = G\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Sedlej:

$$x^1 = G\left[\begin{bmatrix} 2\pi \\ 0 \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x^2 = G\left[\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} \sin\frac{1}{4} \\ \cos\left(\frac{-\sin\frac{1}{4} + 2}{4}\right) \end{bmatrix}$$

$$G(x, y) = \begin{bmatrix} \sin\left(\frac{2x-y}{u}\right) \\ \cos\left(\frac{x+2y}{u}\right) \end{bmatrix}$$

$$J_G(x, y) = \begin{bmatrix} \cos\left(\frac{2x-y}{u}\right) \cdot \frac{1}{2} & -\cos\left(\frac{2x-y}{u}\right) \frac{1}{u} \\ \underbrace{-\sin\left(\frac{x+2y}{u}\right) \cdot \frac{1}{u}}_{|1| < 1} & -\sin\left(\frac{x+2y}{u}\right) \cdot \frac{1}{2} \end{bmatrix}$$

Vermutlich norm

Werte für abs. höchst. Wert:

$$|J_G(x, y)|_{\infty} = \max \left\{ \left| \cos\left(\frac{2x-y}{u}\right) \frac{1}{2} \right| + \left| \cos\left(\frac{x+2y}{u}\right) \right|, \dots \right\} \leq$$

$$\leq \max \left\{ \frac{1}{2} + \frac{1}{u}, \frac{1}{2} + \frac{1}{u} \right\} < 1$$

Es gilt x, y liegt außerhalb aller
Kurve dann es ist möglich

zu zeigen:

$$G(x, y) = \begin{bmatrix} \sin\left(\frac{2x-y}{u}\right) \\ \cos\left(\frac{\sin\left(\frac{2x-y}{u}\right) + 2y}{u}\right) \end{bmatrix}$$

$$J_G = \begin{bmatrix} \frac{1}{2} \cos\left(\frac{2x-y}{u}\right) & -\frac{1}{u} \cos\left(\frac{2x-y}{u}\right) \\ -\frac{1}{2} \cos\left(\frac{2x-y}{u}\right) \sin\left(\frac{\sin\left(\frac{2x-y}{u}\right) + 2y}{u}\right) & \dots \end{bmatrix}$$

Spät mögliches $\| \cdot \|_{\infty}$ norm in vierten
Jacobian < 1

Newtonova metoda

$$x^{(r+1)} = x^{(r)} - J_F(x^{(r)})^{-1} F(x^{(r)})$$

2)

$$J_F(x^{(r)}) (x^{(r+1)} - x^{(r)}) = -F(x^{(r)})$$

1) $J_F(x^{(r)}) \Delta x^{(r)} = -F(x^{(r)})$

2) $x^{(r+1)} = \Delta x^{(r)} + x^{(r)}$

Dan je sistem $x^2 + y^2 = 4$
 $x^2 - y^2 = 1$

a) Naredite 2. korak newtonove metoda
pri $x^{(0)} = (2, 1)$

$$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad F = \begin{bmatrix} x^2 + y^2 - 4 \\ x^2 - y^2 - 1 \end{bmatrix}$$

$$JF = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$$

$$J_F(x^0) \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} 4\bar{x} + 2\bar{y} \\ 4\bar{x} - 2\bar{y} \end{bmatrix} = - \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & | & -1 \\ 4 & -2 & | & -2 \end{bmatrix} \sim \begin{bmatrix} 4 & 2 & | & -1 \\ 0 & -4 & | & 1 \end{bmatrix} \sim y = \frac{1}{4}$$

$$4x + 2y = -1$$

$$4x = -\frac{3}{2}$$

$$x = -\frac{3}{8}$$

3.12

$f_{1. \text{relage}}$

Predločeni sistemi

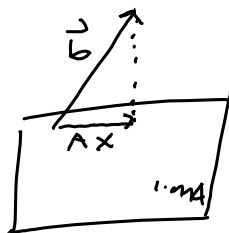
$$Ax = b \quad A \in \mathbb{R}^{m \times n}, \text{ m} \geq n \quad b \in \mathbb{R}^m$$

Iščemo x , ki minimizira $\|Ax - b\|_2$

A polnega range $\Rightarrow$ enolična rešitev

Normalni sistem

$$Ax \in \text{im}(A)$$



$$\Rightarrow Ax - b \perp \text{im}(A)$$

$$\Leftrightarrow \text{postolpek}$$

$$(Ax - b) \perp A^T$$

$$\Leftrightarrow$$

$$A^T(Ax - b) = 0$$

$$\Leftrightarrow$$

$$A^T A x = A^T b$$

$A^T A$ je simetrična, pozitivno definitna
 $\Rightarrow$ lahko uporabimo razcep
choleskega

$$1) A^T A = V V^T$$

$$2) A^T A x = A^T b$$

$$V V^T x = A^T b$$

$$\underbrace{\quad}_y$$

$$V y = A^T b$$

f podana v točkah

$$f(-1) = \frac{11}{4} \quad \text{Poiščite parabolo, ki se}$$

$$f(0) = \frac{7}{4} \quad \text{po metodi najmanjših}$$

$$f(1) = \frac{1}{4} \quad \text{kvadratov najbolj prilega}$$

$$f(2) = \frac{13}{4} \quad \text{funkciji f. Izpeljite normalen}$$

in ga rešite z razcepom Cholskega

$$p(x) = ax^2 + bx + c$$

$$p(-1) = a - b + c = \frac{11}{4}$$

$$p(0) = c = \frac{7}{4}$$

$$p(1) = a + b + c = \frac{1}{4}$$

$$p(2) = 4a + 2b + c = \frac{13}{4}$$

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underline{b}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}$$

Cholesky $V = \begin{bmatrix} 2 & & \\ 1 & \sqrt{5} & \\ 3 & \frac{8-3}{\sqrt{5}} & \sqrt{18-9-5} \end{bmatrix} =$

$$= \begin{bmatrix} 2 & & \\ 1 & \sqrt{5} & \\ 3 & \sqrt{5} & 2 \end{bmatrix} \quad Vy = A^T b$$

$$A^T b = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 11 \\ 7 \\ 1 \\ 13 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 2 & & \\ 1 & \sqrt{5} & \\ 3 & \sqrt{5} & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 16 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= 4 \\ 4 + \sqrt{5}y_2 &= 4 \Rightarrow y_2 = 0 \\ 12 + 2y_3 &= 16 \Rightarrow y_3 = 2 \end{aligned}$$

$$V^T x = y$$

$$\begin{bmatrix} 2 & 1 & 3 \\ & \sqrt{5} & \sqrt{5} \\ & & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \Rightarrow \begin{aligned} x_3 &= 1 \\ \sqrt{5}(x_2 + 1) &= 0 \Rightarrow x_2 = -1 \\ 2x_1 - 1 + 3 &= 4 \\ x_1 &= 1 \end{aligned}$$

$$x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \begin{aligned} a &= 1 \\ b &= -1 \\ c &= 1 \end{aligned}$$

$$\underline{p(x) = x^2 - x + 1}$$

QR razcep

$$A = QR \quad A \in \mathbb{R}^{m \times n} \quad m > n \quad Q \in \mathbb{R}^{m \times n} \quad R \in \mathbb{R}^{n \times n}$$

↑
ortanormirani
stolpci

↑
zagonje
trikotna

$$Q^T Q = I$$

$$Q Q^T \neq I \text{ ker vrstice niso ortanormirane}$$

• Modificiran Gram-Schmidov postopek

$$k = 1, \dots, n$$

$$g_k = a_k$$

$$i = 1, \dots, k-1$$

$$r_{ik} = g_i^T g_k$$

$$g_k = g_k - r_{ik} g_i$$

$$r_{kk} = \|g_k\|_2$$

$$q_k = \frac{g_k}{r_{kk}}$$

S pomočjo MGS (modified gram Schmidt)
izračunajte QR razcep za matriko

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{2} & 0 & -2\sqrt{2} \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 2\sqrt{3} \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$g_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \|g_1\| = \sqrt{2}$$

$$g_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \quad \begin{matrix} r_{12} = 0 \\ g_2 = g_2 - 0 \end{matrix} \quad \begin{matrix} r_{22} = \sqrt{2} \\ g_2 = \frac{g_2}{\sqrt{2}} \end{matrix}$$

$$g_3 = \begin{bmatrix} -1 \\ -3 \\ 1 \\ 1 \end{bmatrix} \quad r_{13} = -\frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}} = -2\sqrt{2}$$

$$g_3 = \begin{bmatrix} -2 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$r_{23} = 0 \quad r_{33} = \sqrt{12} = 2\sqrt{3}$$

$$g_3 = g_3$$

$$g_3 = \frac{1}{2\sqrt{3}} g_3$$

Modificiran Gram-Schmidt

$$k = 1 : n$$

$$z_k = a_k$$

$$j = 1, \dots, k-1$$

$$r_{jk} = z_j^T z_k$$

$$z_k = z_k - r_{jk} z_j$$

$$r_{kk} = \|z_k\|_2$$

$$\hat{z}_k = \frac{1}{r_{kk}} z_k$$

Pri reševanju sistema z MGS naredimo QR razcep na razširjeni matriki

$$[A; b] = [Q; z_{n+1}] \begin{bmatrix} R & z \\ 0 & \rho \end{bmatrix} \quad \left(\begin{array}{l} Ax = b \\ \min_x \|Ax - b\|_2 \end{array} \right)$$

$$\text{Rešjeno } Rx = z \quad \min_x \|Ax - b\| = \rho$$

Od zadnjic

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \quad R = \begin{bmatrix} \sqrt{2} & 0 & -2\sqrt{2} \\ & \sqrt{2} & 0 \\ & & 2 \end{bmatrix}$$

b) Preko QR razcepa rešite predložen sistem
 $Ax = \begin{bmatrix} 4 \\ 6 \\ -1 \\ 2 \end{bmatrix}$ Po metodi najmanjših kvadratov.

koliko je $\min \|Ax - b\|_2$

$$[A; b] = \begin{bmatrix} 1 & 0 & -1 & 4 \\ 1 & 0 & -3 & 6 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \quad z_4 = \begin{bmatrix} 6 \\ 6 \\ -1 \\ 2 \end{bmatrix}$$

$$j=1 \quad r_{14} = \frac{4}{\sqrt{2}} + \frac{6}{\sqrt{2}} = \frac{10}{\sqrt{2}} = 5\sqrt{2}$$

$j=2$:

$$r_{24} = -\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}} = -\frac{3}{\sqrt{2}} = -\frac{\sqrt{2} \cdot 3}{2}$$

$$z_4 = z_4 - 5\sqrt{2} z_1$$

$$z_4 = z_4 + \frac{3}{\sqrt{2}} z_2 = \begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$z_4 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$$

$j=3$

$$r_{34} = \frac{1}{2}$$

$$z_3 = z_4 + \frac{1}{2} z_3 = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$r_{44} = \sqrt{\frac{9}{16} \cdot 2} = \sqrt{\frac{9}{8}} = \frac{3}{2\sqrt{2}}$$

$$z_4 = \frac{1}{r_{44}} z_4 = \frac{1}{2} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Rx = z$$

$$R = \begin{bmatrix} \sqrt{2} & 0 & -2\sqrt{2} \\ & \sqrt{2} & 0 \\ & & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5\sqrt{2} \\ -\frac{3}{2}\sqrt{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$x_3 = -\frac{1}{4}$$

$$x_2 = -\frac{3}{2}$$

$$\sqrt{2} x_1 - 2\sqrt{2} = 5\sqrt{2}$$

$$x_1 = \frac{1}{2} + 5 = \frac{3}{2}$$

$$x = \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \\ -\frac{1}{4} \end{bmatrix}$$

$$\min_x \|Ax - b\|_2 = 2$$

Y

10.12

1.12.2

Razširjeni QR razcep

$$A = \tilde{Q} \tilde{R}$$

$$\tilde{Q} : m \times m \text{ ortogonalna } \tilde{Q}^T \tilde{Q} = \tilde{Q} \tilde{Q}^T = I$$

$$\tilde{R} : m \times n \text{ zgornja trapezna}$$

$$A = \tilde{Q} \tilde{R} = \begin{bmatrix} Q & Q_1 \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix}$$

Reševanje sistema: $\downarrow$ množenje z $\tilde{Q}$ ne spremeni 2-norme

$$\|Ax - b\|_2 = \|\tilde{Q} \tilde{R}x - b\|_2 = \|\tilde{Q}^T (\tilde{Q} \tilde{R}x - b)\|_2 =$$

$$= \|\tilde{R}x - Q^T b\| = \left\| \begin{bmatrix} R \\ 0 \end{bmatrix} x - \begin{bmatrix} Q^T \\ Q_1^T \end{bmatrix} b \right\|$$

x , ki to minimizira dobimo iz $Rx = Q^T b$

$$\text{minimum} = \|Q_1^T b\|_2$$

Givensove rotacije

vhodni podatki: A, b

izhod: $\tilde{R} \tilde{Q}^T b$

for $j=1:n$

for $k=i+1:m$

if $a_{ki} \neq 0$:

$$R_{ik}^T = I$$

$$r = \sqrt{a_{ii}^2 + a_{ki}^2}$$

$$c = \frac{a_{ii}}{r}$$

$$s = \frac{a_{ki}}{r}$$

$$(R_{ik}^T)_{ii} = c$$

$$(R_{ik}^T)_{ik} = s$$

$$R_{ik}^T([i \ k], [i \ k]) = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

$$A = R_{ik}^T A$$

$$b = R_{ik}^T b$$

izhod: $A (= \tilde{R})$, $b (= \tilde{Q}^T b)$

Naloga: S pomočjo givensovih rotacij
reši predano sistem. koliko je $\min \|Ax - b\|$?

$$\begin{bmatrix} 3 \\ -4 \\ 12 \\ -84 \end{bmatrix} x = \begin{bmatrix} 7 \\ -1 \\ 5 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 85 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ 5 \\ -7.5 \\ 84 \\ \hline 80 \end{bmatrix}$$

$i=1 \quad k=2$

$$\begin{aligned} r &= 5 \\ c &= \frac{3}{5} \\ s &= -\frac{4}{5} \end{aligned} \quad R = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & & \\ \frac{4}{5} & \frac{3}{5} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \quad \begin{array}{l} RA \uparrow \\ RB \uparrow \end{array}$$

$$\frac{144}{25} \\ \hline 169$$

$i=1 \quad k=3$

$$r = \sqrt{5^2 + 12^2} = 13$$

$$c = \frac{5}{13} \quad s = \frac{12}{13}$$

$$R = \begin{bmatrix} \frac{5}{13} & 0 & \frac{12}{13} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{12}{13} & 0 & \frac{5}{13} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$RA \uparrow$
 $RB \uparrow$

$i=1 \quad k=4$

$$r = \sqrt{13^2 + 84^2} = 85$$

$$c = \frac{13}{85} \quad s = -\frac{84}{85}$$

$$R = \begin{bmatrix} \frac{13}{85} & & & -\frac{84}{85} \\ & 1 & & \\ & & 1 & \\ \frac{84}{85} & & & \frac{13}{85} \end{bmatrix}$$

$$A = \begin{bmatrix} 85 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 5 \\ -\frac{35}{13} \\ \frac{84}{13} \end{bmatrix}$$

$$Q = 1$$

$$\rightsquigarrow R = 85$$

$$Q_1^T b = \begin{bmatrix} 5 \\ -\frac{35}{13} \\ \frac{84}{13} \end{bmatrix}$$

$$\text{minimum: } \sqrt{25 + \frac{35^2}{13^2} + \frac{84^2}{13^2}} \\ = \sqrt{74}$$

X mankarije 1,5 krat

7 dno pre
in
24.12

Interpolacija

Iščemo polinom, ki bo potekel skozi dane točke. Za $n+1$ takh obstaja edinstven polinom stopnje n , ki jih interpolira

Lagrangeova oblika interpolacijskega polinoma

Lagrangevi: bazni polinomi:

$$l_{i,n}(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{(x-x_j)}{(x_i-x_j)}$$

interpolacijski polinom:

$$p(x) = \sum_{i=0}^n f(x_i) l_{i,n}(x)$$

Dobcitz L.I.P.

$f(x) = \frac{40}{x+1}$ v. t durch $\{x_0, x_1, x_2, x_3\}$

$$n=3$$

$$l_0 = \frac{x - \frac{1}{3}}{-\frac{1}{3}} \cdot \frac{x - \frac{2}{3}}{-\frac{2}{3}} \cdot \frac{x - 1}{-1} = -(3x - 1)\left(\frac{3}{2}x - 1\right)(x - 1)$$

$$l_1 = \frac{x}{\frac{1}{3}} \cdot \frac{x - \frac{2}{3}}{-\frac{1}{3}} \cdot \frac{x-1}{-\frac{2}{3}} \quad + u_0 = f(x_i)$$

$$b_2 = \frac{x}{\frac{2}{3}} \cdot \frac{x - \frac{1}{3}}{\frac{1}{3}} \cdot \frac{x - 1}{-\frac{1}{3}} \quad f(x_2) = 24$$

$$l_3 = \frac{x}{1} \cdot \frac{x - \frac{1}{3}}{\frac{2}{3}} \cdot \frac{x - \frac{2}{3}}{\frac{1}{3}} \quad f'(x_3) = 20$$

$$\begin{aligned}
 p(x) &= \frac{3}{2}(x - \frac{2}{3})(x-1)(3x-1)(40+30x) + \frac{60}{24} \\
 &\quad + \frac{9}{2}x(x - \frac{1}{3})(24(x-1) + 60(x - \frac{2}{3})) = \\
 &= \frac{6}{2}(3x-2)(x-1)(-30x+40) + \frac{40}{-24} \\
 &\quad \frac{6}{2}(3x^2-1)(36x-16) = \frac{16}{16} \\
 &= \frac{6}{2}(3x-4)(-10(3x^2-5x+2) + 12x^2-4) = \\
 &= \frac{6}{2}(3x-4)(-18x^2-50x-24) = \\
 &= 6(3x-4)(-9x^2-25x-12) = \\
 &= 6(x^3(-81) + x^2(36) + x(-9 \cdot 12 + 4 \cdot 25) + 12 \cdot 4)
 \end{aligned}$$

Newtonova oblika interpolacijskega polinoma

Baza iz premaknjenih potenc

$$1, x-x_0, (x-x_0)(x-x_1), \dots, (x-x_0)\dots(x-x_{n-1})$$

$$p(x) = \sum_{i=0}^n \underbrace{[x_0, x_1, \dots, x_i]}_{\text{deljene difference}} f(x-x_0)\dots(x-x_{i-1})$$

$$[x_0 \dots x_k] f = \frac{\overset{\substack{\uparrow \\ n_i \text{ male}}}{[x_1 \dots x_k] f} - [x_0 \dots x_{k-1}] f}{x_k - x_0}$$

x_i	$[x_i]$	$[,] f$	$L_i,] f$
0	40	$\frac{30-40}{\frac{1}{3}} = -30$	$\frac{18}{\frac{2}{3}} = 18$
$\frac{1}{3}$	30	$\frac{24-30}{\frac{1}{3}} = -18$	$\frac{9-18}{1} = -9$
$\frac{2}{3}$	24	$\frac{20-24}{\frac{1}{3}} = -12$	
1	20		

$$p(x) = 40 - 30(x-0) + 18x(x-\frac{1}{3}) - 9x(x-\frac{1}{3})(x-\frac{2}{3})$$

$$40 - 30x + 18x^2 - 6x^3 + 9x^2 - 2 =$$

$$= 40 - 38x + 27x^2 - 6x^3$$

Vemo s prelevanj

$$p(x) - p(x) = w(x) \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$w(x) = \prod_{i=1}^n (x-x_i)$$

$$w(x) = x(x-\frac{1}{3})(x-\frac{2}{3})(x-1)$$

Nyborz om
menjkele

Poljane difference

$$[X_i, \dots, X_{i+k}]f = \begin{cases} \frac{f^{(k)}(x_i)}{k!} & ; x_i = \dots = x_{i+k} \\ \frac{[X_{i+1}, \dots, X_{i+k}]f - [X_i, \dots, X_{i+k-1}]f}{X_{i+k} - X_i} & ; \text{si } x_i \neq x_{i+k} \end{cases}$$

$$f(x) - p(x) = \omega(x) \frac{f^{(n+1)}(\xi)}{(n+1)!} \quad \omega(x) = \prod_{i=0}^n (x - x_i)$$

$$f(x) - p(x) = \omega(x) [x_0, x_1, \dots, x_n, x]$$

Mej bo $f(x) = \frac{40}{x+1}$

a) Določite polinom p , ki interpolira vrednosti in odvode funkcije f v točkah 0 in 1 neke oceniti

napako $\|f-p\|_{\infty, [0,1]}$

$p(0) = 40$

$p(1) = 20$

$p'(0) = -40$

$p'(1) = -10$

$f' = -\frac{40}{(x+1)^2}$

$f(0)$

	$[0]f$	$[0, \cdot]f$	$[0, \cdot, \cdot]f$
x_0	0	40	
x_1	0	40	$-\frac{40}{1^2} = \frac{f'(0)}{1!}$
x_2	1	20	$\frac{20-40}{1-0} = -20$
x_3	1	20	-10

$p(x) = 40 - 40x + 20x^2 - 10x^2(x-1)$

$n=3$ stopnja polinoma

Torej rabimo 4. odvod

$\|f-p\|_{\infty, [0,1]} \leq \|w\|_{\infty, [0,1]} \frac{\|f^{(n+1)}\|_{\infty, [0,1]}}{(n+1)!}$

$f^{(4)}(x) = \frac{80}{(x+1)^3}$

$f^{(4)}(x) = \frac{2 \cdot 3 \cdot 4 \cdot 40}{(x+1)^5}$

$f^{(4)}(x) = -\frac{2 \cdot 3 \cdot 40}{(x+1)^4}$

največje bo pri $x=0$
 $\Rightarrow \|f^{(4)}\|_{\infty} = 2 \cdot 3 \cdot 4 \cdot 40$

$\|f-p\|_{\infty} \leq \|w\| \frac{4! \cdot 40}{4!}$

$\|w\| = ?$ $w(x) = x^2(x-1)^2$

$w'(x) = 2x(x-1)^2 + 2x^2(x-1) =$

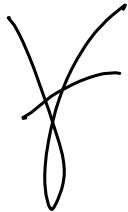
$= 2x(x-1)(2x-1)$

na $0, 1, \frac{1}{2}$

$\Rightarrow \|w\| = w\left(\frac{1}{2}\right) = \frac{1}{16}$

$\|f-p\|_{\infty} \leq 40 \cdot \frac{1}{16} = \frac{5}{2}$

$$\begin{array}{r}
 01100001 \\
 01100010 \\
 \hline
 11000011
 \end{array}$$


 nizam
 padurath

f 1, w =

p = gley si vse
pa Fil:om

Gaussova pravila (red $2n+1$)

Nerazneške so vozli in uteži

Vozle določimo iz pogojev:

$$w \perp 1 \quad w \perp x, \dots \quad w \perp x^n$$

$$w(x) = (x-x_0) \dots (x-x_n)$$

$$\langle g, h \rangle = \int_a^b g(x)h(x) dx$$

Ko poznamo vozle je določen uteži linearen problem

(MNK ali preko int. polinoma)

Za $2n+2$ krat zvezno odvedljivo funkcijo na $[a, b]$ je

napako oblike
$$\frac{f^{(2n+2)}(\xi)}{(2n+2)!} \int_a^b (w(x))^2 dx$$

Dedurrete uteri A_0 in A_1 ter uorde x_0 in x_1
 u Gaussiam pravila

1

$$\int_{-1}^1 f(x) dx = A_0 f(x_0) + A_1 f(x_1) + R(f)$$

Dedurrete hdi repoko $R(f)$

$$w(x) = (x-x_0)(x-x_1)(x-x_2) =$$

$$= (x^2 - (x_0+x_1)x + x_0x_1)(x-x_2) =$$

$$I_0: \int_{-1}^1 w(x) dx = 0$$

$$= x^3 - x^2(x_0+x_1+x_2) + x(x_0+x_1-x_2x_0-x_1x_2) + x_0x_1x_2$$

$$\int_{-1}^1 x w(x) dx = 0$$

$$I_1 = \frac{1}{4}x^4 - \frac{1}{3}x^3(x_0+x_1+x_2) + \frac{1}{2}x^2(x_0+x_1-x_2x_0-x_1x_2) + x x_0x_1x_2 \Big|_{-1}^1 = 0$$

$$\Rightarrow -\frac{2}{3}(x_0+x_1+x_2) + 2x_0x_1x_2 = 0$$

$$I_2 = \frac{1}{5}x^5 - \frac{1}{4}x^4(x_0+x_1+x_2) + \frac{1}{3}x^3(x_0+x_1-x_2x_0-x_1x_2) + \frac{1}{2}x^2x_0x_1x_2 \Big|_{-1}^1 = 0$$

$$= \frac{1}{5} + \frac{2}{3}(x_0+x_1-x_2x_0-x_1x_2) = 0$$

$$x_0+x_1+x_2 = 3x_0x_1x_2$$

$$x_0+x_1-x_2x_0-x_1x_2 = -\frac{3}{5} \quad \left. \begin{array}{l} \\ \end{array} \right\} -$$

$$x_2+x_2x_0+x_1x_2 = 3(x_0x_1x_2 + \frac{1}{5})$$

$$x_2(1+x_0+x_1) = 3x_0x_1x_2 + \frac{3}{5}$$

$$\begin{matrix} 2 & 2 & 2 \\ \underline{2} & \underline{2} & \underline{2} \\ & & 1 \end{matrix}$$

u mo ene step 20
 : glave

$$x_0 = -x_1$$

$$x_1 = \frac{\sqrt{3}}{3}$$

1

$$1: \int_{-1}^1 1 dx = 1 - (-1) = 2 = A_0 \cdot 1 + A_2 \cdot 1$$

$$x: \int_{-1}^1 x dx = \frac{1}{2}x^2 \Big|_{-1}^1 = 0 = A_0x_0 + A_1x_1 = 0$$

$$\text{dobimo } A_0 = 1 \quad A_1 = 1$$

išene repoko

1